# Chapter 3

## RELATIVE INJECTIVITY

# 3. RELATIVE INJECTIVITY

In this chapter we discuss relative injectivity and injectivity of N-groups. This chapter has four sections.

## 3.1 PRELIMINARIES:

This section deals with some basic definitions and results which are used in the later sections.

**Definition 3.1.1:** Let E be an N-group. Then the singular subset of E is defined as the set

$Z(E) = \{ x \in E \,/\, Ix = 0$ for some essential N-subgroup I of N$\}$.

An N-group E is called singular N-group if $Z(E) = E$.

An N-group E is called non-singular N-group if $Z(E) = 0$.

**Definition 3.1.2:** If E is an N-group, the set $Z_w(E) = \{ x \in E \,/\, Ix = 0$ for some essential ideal I of N$\}$ is weak singular subset of E.

An N-group E is called weak singular if $Z_w(E) = E$.

An N-group E is called weak non-singular if $Z_w(E) = 0$.

**Example 3.1.3:** $N = Z_8$ is a near-ring with two operations '+' as addition modulo 8 and '.' defined by following table:

| . | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 2 | 0 | 4 | 4 | 2 |
| 2 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 4 |
| 3 | 0 | 0 | 0 | 6 | 0 | 4 | 4 | 6 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 2 | 0 | 4 | 4 | 2 |
| 6 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 4 |
| 7 | 0 | 0 | 0 | 6 | 0 | 4 | 4 | 6 |

Here $I = \{0, 4\}$ is an essential N-subgroup of N. Here $\forall\, x \in N$, $Ix = 0$. So $Z(N) = N$, so N is singular.

But $I = \{0, 4\}$ is also an essential ideal of N. Hence $Z_w(N) = N$ and so N is also weak singular.

Example 2.1.13 is an example of non-singular as well as weak non-singular N-group.

**Definition 3.1.4:** An N-monomorphism $f : A \rightarrow B$ is said to be an essential N-monomorphism if $fA \leq_e B$.

**Proposition 3.1.5:** An N-group C is singular if there exists a short exact sequence

$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ such that f is an essential N-monomorphism.

**Proof:** Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence such that f is an essential N-monomorphism. For any $b \in B$, we have a map $k : N \to B$ defined by $k(n) = nb$. By proposition 1.3.5, $k^{-1}(fA) \leq_e N$.

$\Rightarrow$ the N-subgroup $I = \{ n \in N / nb \in fA\}$ is an essential N-subgroup of N.

Now $Ib \leq fA = \text{Ker}g$.

Hence $g(Ib) = 0 \Rightarrow I(gb) = 0$ and so $gb \in Z(C)$.

Since g is an N-epimorphism, we get $Z(C) = C \Rightarrow C$ is singular.

**Corollary 3.1.6:** If A is an essential ideal of B, then B/A is singular.

**Proof:** We consider the short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{g} B/A \to 0$.

As $A \leq_e B$, from above proposition B/A is singular.

**Proposition 3.1.7:** If B is Non-singular and B/A is singular then $A \leq_{we} B$.

**Proof:** If B/A is singular and x is non-zero element of B, then $I\bar{x} = \bar{0}$ for some essential N-subgroup I of $N \Rightarrow Ix \leq A$. As B is non-singular, we have $Ix \neq 0$ and thus $Nx \cap A \neq 0$.

Therefore $A \leq_{we} B$.

**Proposition 3.1.8:** If N is a dgnr and $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of N-group E then E is a homomorphic image of $\oplus_{e \in E} Ne$.

**Proof:** Let $f_e$: $Ne \to E$ be defined by $f_e(ne) = ne$.

Then $f_e$ is N-homomorphism.

Let $f_{e_i} : Ne_i \to E$ be defined by $f_{e_i}(n_ie_i) = n_ie_i$ and $f_{e_j} : Ne_j \to E$ be defined by $f_{e_j}(n_je_j) = n_je_j$

Let $f_{e_i} + f_{e_j} : Ne_i \oplus Ne_j \to E$ be defined by $(f_{e_i} + f_{e_j})(n_ie_i + n_je_j) = (f_{e_i}(n_ie_i) + f_{e_j}(n_je_j)$.

Obviously it is well-defined.

Let $(n_i^/e_i + n_j^/e_j)$, $(n_i^{//}e_i + n_j^{//}e_j) \in Ne_i \oplus Ne_j$ and $(f_{e_i} + f_{e_j})((n_i^/e_i + n_j^/e_j) + (n_i^{//}e_i + n_j^{//}e_j))$

$= (f_{e_i} + f_{e_j})((n_i^/e_i + n_i^{//}e_i) + (n_j^/e_j + n_j^{//}e_j))$ [since Ne's are normal N-subgroups ]

$= (f_{e_i} + f_{e_j})((n_i^/ + n_i^{//})e_i) + ((n_j^/ + n_j^{//})e_j))$

$= (n_i^/ + n_i^{//})e_i + (n_j^/ + n_j^{//})e_j$

$= ((n_i^/e_i + n_i^{//}e_i) + (n_j^/e_j + n_j^{//}e_j))$

$= ((n_i^/e_i + n_j^/e_j) + (n_i^{//}e_i + n_j^{//}e_j))$

$= f_{e_i}(n_i^/e_i) + f_{e_j}(n_j^/e_j) + f_{e_i}(n_i^{//}e_i) + f_{e_j}(n_j^{//}e_j))$

$=(f_{e_i} + f_{e_j})(n_i^/e_i + n_j^/e_j) + (f_{e_i} + f_{e_j})(n_i^{//}e_i + n_j^{//}e_j)$

Next for $n \in N$, $(f_{e_i} + f_{e_j})(n(n_i^/e_i + n_i^{//}e_i)) = (f_{e_i} + f_{e_j})(\sum_{i=1}^{n} s_i\ (n_i^/e_i + n_i^{//}e_i))$ [since N dgnr]

$= (f_{e_i} + f_{e_j})(\ s_1(n_i^/e_i + n_i^{//}e_i) + s_2(n_i^/e_i + n_i^{//}e_i) + \ldots + s_n(n_i^/e_i + n_i^{//}e_i))$

$= (f_{e_i} + f_{e_j})((\ s_1n_i^/ + s_2n_i^/ + \ldots + s_nn_i^/)\ e_i + (\ s_1n_i^{//} + s_2n_i^{//} + \ldots + s_nn_i^{//})\ e_j)$

$= (f_{e_i} + f_{e_j})((\ \sum_{i=1}^{n} s_i)\ n_i^/)\ e_i + (\ \sum_{i=1}^{n} s_i)\ n_i^{//})\ e_j$

$=(\ f_{e_i} + f_{e_j})((nn_i^/)e_i + (nn_i^{//})e_j)$

$= (nn_i^/)e_i + (nn_i^{//})e_j$

$= (\sum_{i=1}^{n} s_i)\, n_i^{/}e_i + (\sum_{i=1}^{n} s_i)\, n_i^{//}e_j$

$= ((s_1n_i^{/}e_i + s_2n_i^{/}e_i + \ldots + s_nn_i^{/}e_i) + (s_1n_i^{//}e_j + s_2n_i^{//}e_j + \ldots + s_nn_i^{//}e_j))$

$=(s_1(n_i^{/}e_i + n_i^{//}e_i) + s_2(n_i^{/}e_i + n_i^{//}e_i) + \ldots + s_n(n_i^{/}e_i + n_i^{//}e_i))$

$= (\sum_{i=1}^{n} s_i\, (n_i^{/}e_i + n_i^{//}e_i))$

$= n(f_{e_i} + f_{e_j})\, (n_i^{/}e_i + n_i^{//}e_i)$

Thus $(f_{e_i} + f_{e_j})$ is an N-homomorphism.

Similarly if we define $f = \Sigma_{e\in E} f_e : \oplus_{e\in E} Ne \rightarrow E$ by $(\Sigma_{e\in E} f_e)\,(\Sigma_{e\in E}\, ne) = (\Sigma_{e\in E}\, f_e(ne))$ , n $\in$ N, it is an N-homomorphism.

Obviously it is an N-monomorphism.

Again for any $e_k \in E$ we get $e_k \in Ne_k \in \oplus_{e\in E} Ne$. So f is onto.

Hence E is a homomorphic image of $\oplus_{e\in E} Ne$.

**Theorem 3.1.9:** For a short exact sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ if A and C are finitely generated then B is also finitely generated.

**Proof**: As $\beta : B \rightarrow C$ is an epimorphism, $C \cong \frac{B}{Ker\beta} \Rightarrow C \cong \frac{B}{\alpha(A)}$.

For identity map $\alpha$, $C \cong \frac{B}{A}$.

So if an N-group B has finitely generated N-subgroup A and factor N-group $\frac{B}{A}$ then B is also finitely generated.

**Definition 3.1.10:** For an N-group E an element x is called a nilpotent element if $x^k = 0$ for some $k \in \Gamma^+$.

## 3.2 E-injectivity and injectivity:

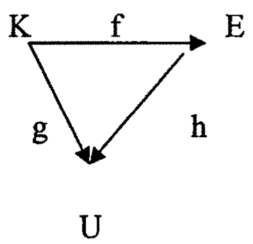
In this section we define relative injective N-groups, and some special relative injective N-groups and investigate various characteristics of these N-groups.

In the third section of the chapter we study direct sums of relative injective N-groups and N-subgroups, direct product of relative injective N-groups. Using the notion of dominance of an element of an N-group by another N-group direct sums of relative injective N-groups are established.

In the last section we are trying to relate direct sums of relative injective N-groups and chain conditions, relative injectivity of simple, semi-simple, strictly semi-simple , singular N-groups and chain conditions.

**Throughout the remaining section of this chapter we consider all N-groups unitary N-groups unless otherwise specified.**

**Definition 3.2.1:** Let E and U be N-groups. U is called E- injective or U is injective relative to E if for each N-monomorphism $f : K \rightarrow E$, every N –homomorphism from K into U can be extended to an N- homomorphism from E into U. i.e. The diagram



commutes. i.e. $g = hf$.

An N-group A is injective if it is E-injectve for every N-group E of N. So if an N-group A is injective it is E-injectve for any N-group E.

**Proposition 3.2.2:** Let N be a dgnr, E be an N-group and F be a commutative N-group. Then the set $Hom_N(E, F) = \{ f / f : E \rightarrow F$ is an N-homomorphism$\}$ is an abelian group where addition is defined as : for $f, g \in Hom_N(E, F)$, $( f + g )(e) = f(e) + g(e)$.

**Proof:** As F is an abelian N-group, for $f, g \in Hom_N(E, F)$ and $e \in E$,

$( f + g)(e) = f(e) + g(e)$

$= g(e) + f(e)$

$= ( g + f )(e)$, so $f + g = g + f$.

We are to show $f + g$ is an N-homomorphism.

For $e_1, e_2 \in E$, $( f + g )(e_1 + e_2) = f(e_1 + e_2) + g(e_1 + e_2)$ [ By given condition]

$= f(e_1) + f(e_2) + g(e_1) + g(e_2)$ [$\because$ f,g are N-homomorphism]

$= f(e_1) + g(e_1) + f(e_2) + g(e_2)$ [$\because$ F is abelian]

$= (f + g)(e_1) + ( f + g)(e_2)$ [ By given condition]

Next for $e \in E$, $n \in N$

$( f + g )(ne) = f(ne) + g(ne)$ [ By given condition]

$= nf(e)+ ng(e)$ [$\because$ f, g are N-homomorphisms]

$= ( \sum_{i=1}^{n} s_i)f(e) + ( \sum_{i=1}^{n} s_i)g(e)$ [$\because$ N is dgnr]

$= s_1f(e) + s_2f(e) + \ldots + s_nf(e) + s_1g(e) + s_2g(e) + \ldots + s_ng(e)$

$= s_1f(e) + s_1g(e) + s_2f(e) + s_2g(e) + \ldots + s_nf(e) + s_ng(e)$ $[\because\ s_if(e), s_ig(e) \in F]$

$= s_1 (f(e) + g(e)) + s_2 (f(e) + g(e)) + \ldots + s_n (f(e) + g(e))$

$= s_1 ((f + g)(e)) + s_2((f + g)(e)) + \ldots + s_n ((f + g)(e))$

$= (s_1 + s_2 + \ldots + s_n) ((f + g)(e))$

$= n((f + g)(e))$

Thus f + g is an N-homomorphism.

**Proposition 3.2.3:** Let B, M be two N-groups and C an ideal of B. For N-homomorphism

$f : B \to M$ $\exists$ unique homomorphism $\bar{f} : \frac{B}{C} \to M$ such that $\bar{f}(\bar{b}) = f(b)$, $\forall$ $C \subseteq Kerf$.

**Proof:** Let $\overline{b_1} = \overline{b_2}$

$\Rightarrow \overline{b_1} - \overline{b_2} = \bar{0}$

$\Rightarrow b_1 - b_2 + C = C$

$\Rightarrow b_1 - b_2 \in C \subseteq Kerf$

$\Rightarrow f(b_1 - b_2) = 0$

$\Rightarrow f(b_1) - f(b_2) = 0$

$\Rightarrow \bar{f}(\overline{b_1}) = \bar{f}(\overline{b_2})$

So $\bar{f}$ is well-defined.

Next $\bar{f}(\overline{b_1} + \overline{b_2})$

$= \bar{f}(\overline{b_1 + b_2})$

$= f(b_1 + b_2)$

$= f(b_1) + f(b_2)$

$= \bar{f}(\overline{b_1}) + \bar{f}(\overline{b_2})$

And $\bar{f}(n\,\bar{b})$

$= \bar{f}(n(b + C))$

$= \bar{f}(nb + C)$

$= \bar{f}(\overline{nb})$

$= f(nb)$

$= nf(b)$

$= n\bar{f}(\bar{b})$

So $\bar{f}$ is an N-homomorphism and by definition obviously it is unique.

**Thus we get if f is an epimorphism, then $\bar{f}$ defined as above is also an epimorphism.**

**Definition 3.2.4:** Let U be an commutative N-group and $f : L \to M$ be an N-homomorphism. We can define a mapping

$f^* = \text{Hom}_N(f, U) : \text{Hom}_N(M, U) \to \text{Hom}_N(L, U)$

by $\text{Hom}_N(f, U) : \gamma \to \gamma f$ i.e. $f^*\gamma = \gamma f$ then $\text{Hom}_N(f, U)$ is an N-homomorphism..

**Proposition 3.2.5:** If U is a commutative N-group, then for every exact sequence

$$0 \to K \xrightarrow{f} E \xrightarrow{g} L \to 0$$

the sequence $0 \to \text{Hom}_N(L, U) \xrightarrow{g^*} \text{Hom}_N(E, U) \xrightarrow{f^*} \text{Hom}_N(K, U)$ is exact.

**Proof:** If $\gamma \in \text{Hom}_N(L, U)$ and $g^*(\gamma) = 0$

$\Rightarrow \gamma g = 0$

$\Rightarrow \gamma = 0 \quad [\because g$ is N-epimorphism]

$\Rightarrow g^*$ is N-monomorphism.

Next let $\gamma \in Hom_N(L,U)$. Then $f^* g^*(\gamma) = f^*(\gamma g) = (\gamma g)f = \gamma(gf) = \gamma 0 = 0^* = 0 = 0\gamma$

So we get $f^* g^* = 0 \Rightarrow im\, g^* \subseteq Ker\, f^*$.

Next let $\beta \in Ker\, f^*$, then $\beta f = f^*\beta = 0$

$\Rightarrow \beta(imf) = 0 \Rightarrow \beta(Kerg) = 0$

$\Rightarrow Kerg \subseteq Ker\beta$.

Now $\beta : E \to U$ is an N-homomorphism such that $Kerg \subseteq Ker\beta$.

$\Rightarrow \exists$ a unique N-homomorphism $\bar{\beta} : \frac{E}{Kerg} \to U$ such that $\bar{\beta}(\bar{b}) = \beta(b)$.

Also g: $E \to L$ is an N-epimorphism, so $\exists$ an N-isomorphism $\phi : \frac{E}{Kerg} \to L$ such that

$\phi(\bar{b}) = g(b)$.

We consider the following sequence of N-homomorphisms

$L \xrightarrow{\phi^{-1}} \frac{E}{Kerg} \xrightarrow{\bar{\beta}} U$, which gives $\bar{\beta}\, \phi^{-1} \in Hom_N(L, U)$.

Now $g^*(\bar{\beta}\, \phi^{-1}) = (\bar{\beta}\, \phi^{-1})g = \beta$

$\Rightarrow \beta \in im\, g^*$. [since $g^*(\bar{\beta}\, \phi^{-1})(b) = ((\bar{\beta}\, \phi^{-1})g)(b) = \bar{\beta}(\bar{b}) = \beta(b)$].

So $img^* = Ker\, f^*$.

**Proposition 3.2.6:** A commutative N-group U is E-injective if and only if $Hom_N(-, U)$ is exact.

**Proof:** We assume U is E-injective.

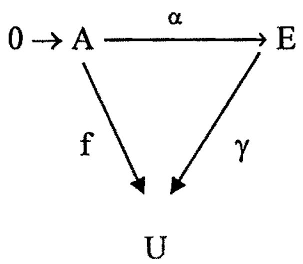
We consider the exact sequence $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} C \to 0$.

Now exactness of $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} C \to 0$ implies

$0 \to Hom_N(C, U) \xrightarrow{\beta^*} Hom_N(E, U) \xrightarrow{\alpha^*} Hom_N(A,U)$ is exact.

So it is enough to show $\alpha^*$ is epic.

Let $f \in Hom_N(A, U)$. We consider the diagram

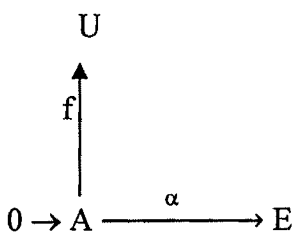



Since U is injective, $\exists\ \gamma \in Hom_N(E, U)$ such that $\gamma\alpha = f$

$\Rightarrow \alpha^*\gamma = f$

$\Rightarrow \alpha^*$ is onto.

Conversely, let HomN( - , U) be exact. We consider the diagram with exact row



$$0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \frac{E}{im\alpha} \to 0 \text{ is exact.}$$

$$\Rightarrow 0 \to Hom_N(\frac{E}{im\alpha}, U) \xrightarrow{\beta^*} Hom_N(E, U) \xrightarrow{\alpha^*} Hom_N(A, U) \to 0 \text{ is exact.}$$

Since $\alpha^*$ is an epimorphism, for $f \in Hom_N(A, U)$ such that $\alpha^*\gamma = f$

$$\Rightarrow \gamma\alpha = f.$$

Thus $\exists\, \gamma : E \to U$ such that $\gamma\alpha = f \Rightarrow U$ is E- injective.

**Definitions 3.2.7:** An N-group E is a WI-N-group if N-group W is E-injective.

**Definition 3.2.8:** An N-group E is a $W_C$I-N-group if a commutative N-group W is E-injective.

**Definition 3.2.9:** An N-group E is called a s-simple or a strict simple N-group if it has no proper normal N-subgroups.

Proposition 1.3.12 holds for normal N-subgroups also. Thus we get the following proposition:

**Proposition 3.2.10:** The following are equivalent

(a) Every normal N-subgroup of E is a direct summand.

(b) E is a sum of simple normal N-subgroups.

(c) E is a direct sum of simple normal N-subgroups.

**Definitions 3.2.11:** We define s-Soc E or strict socle of E as direct sum of simple normal N-subgroups.

An N-group E is called a strictly semisimple N-group if s-Soc(E) = E. In other words E is strictly semisimple if one of the conditions of proposition 3.2.10 holds.

We observe that every semisimple N-group is strictly semisimple but the converse is not true. If N is a dgnr then every strictly semisimple N-group is semisimple.

The following is an example of strictly semisimple N-group which is not semisimple.

**Example 3.2.12:** We consider the near-ring N = { 0, a, b, c, x, y } under the addition and multiplication defined as the following table

| + | 0 | a | b | c | x | y |
|---|---|---|---|---|---|---|
| 0 | 0 | a | b | c | x | y |
| a | a | 0 | y | x | c | b |
| b | b | x | 0 | y | a | c |
| c | c | y | x | 0 | b | a |
| x | x | b | c | a | y | 0 |
| y | y | c | a | b | 0 | x |

| . | 0 | a | b | c | x | y |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c | 0 | 0 |
| b | 0 | a | b | c | 0 | 0 |
| c | 0 | a | b | c | 0 | 0 |
| x | 0 | 0 | 0 | 0 | 0 | 0 |
| y | 0 | 0 | 0 | 0 | 0 | 0 |

Here { 0, a }, { 0, b }, { 0, c }, { 0, x, y } are simple left normal N-subgroups of N.

And N = { 0, a } + { 0, b } + { 0, c } + { 0, x, y }. So N is strictly semisimple.

But N is not semisimple.

**Definitions 3.2.13:** An N-group E is called SI N-group if every singular N-group is E-injective.

An N-group E is called $S_wI$ N-group if every weak singular N-group is E-injective.

An N-group E is called V N-group if every simple N-group is E-injective.

An N-group E is called $V_c$ N-group if every simple commutative N-group is E-injective.

An N-group E is called GV N-group if every simple singular N-group is E-injective.

An N-group E is called $S^2$ I N-group if every strictly semi-simple N-group is E-injective.

An N-group E is called $S^3$ I N-group if every strictly semi-simple singular N-group is E-injective.

An N-group E is called $S^2S_wI$ N-group if every strictly semi-simple weak singular N-group is E-injective.

**Definition 3.2.14:** A near-ring N is called V near-ring if $_NN$ is a V N-group and GV near-ring if $_NN$ is a GV N-group.

A near-ring N is called $V_c$ near-ring if $_NN$ is a $V_c$ N-group.

**Proposition 3.2.15:** N-subgroups of a WI N-group are again WI N-groups.

**Proof:** Let E be a WI N-group.

$\Rightarrow$ W is E-injective.

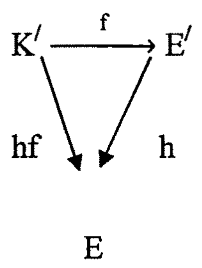
And let $E'$ be any N-subgroup of E.

We show $E'$ is also a WI N-group.

That is we are to show W is also $E'$-injective.

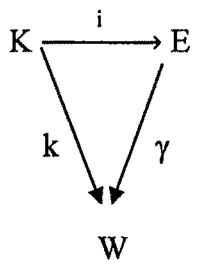
Let $h : E' \to E$ be an N-monomorphism and $K'$ be an N-subgroup of $E'$ and $f : K' \to E'$ be any N-monomorphism.

Then hf is also an N-monomorphism, $hf : K' \to E$.




Now W is E-injective, so for any N-subgroup K of E, the N-monomorphism $i : K \to E$ and any N-homomorphism $k : K \to W$, $\exists$ an N- homomorphism $\gamma : E \to W$ s.t. $k = \gamma i$.

i.e. the following diagram

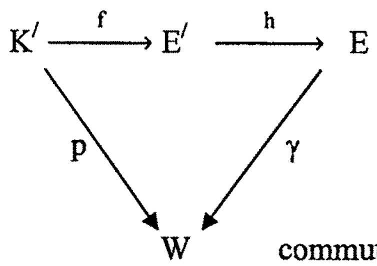



commutes.

Since W is E-injective, so for N- monomorphism $hf : K' \to E$ and $p : K' \to W$ we get

$\gamma : E \rightarrow W$ such that $\gamma(hf) = p$.

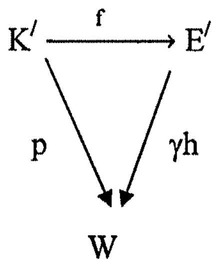
That is the diagram




commutes.

Now $f : K' \rightarrow E'$ is an N-monomorphism and for any N- homomorphism $p : K' \rightarrow W$,

we get $\gamma h : E' \rightarrow W$ such that the diagram




commutes. That is $p = (\gamma h)f$.

Therefore W is $E'$- injective.

**Proposition 3.2.16:** Homomorphic images of a $W_C$I N-groups are again $W_C$I N-groups.

**Proof:** Given $0 \rightarrow E' \xrightarrow{h} E \xrightarrow{k} E'' \rightarrow 0$ is exact and commutative N-group W is E-injective.

We show W is $E''$-injective.

Let $E' \leq K \leq E$ and that $E'' = E/E'$. Now we consider the canonical diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E/E' & \longrightarrow & 0 \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\
0 & \longrightarrow & E' & \longrightarrow & K & \longrightarrow & K/E' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now applying $Hom_N(\text{-}, W)$ we get the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Hom_N(E/E', W) & \rightarrow & Hom_N(E, W) & \rightarrow & Hom_N(E', W) & \rightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \rightarrow & Hom_N(K/E', W) & \rightarrow & Hom_N(K, W) & \rightarrow & Hom_N(E', W) & \rightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Since $Hom_N(E/E', W) \xrightarrow{\phi} Hom_N(K/E', W)$ is epic, for all $\gamma \in Hom_N(K/E', W)$ $\exists$ $\alpha \in$ $Hom_N(E/E', W)$ such that $\phi(\alpha) = \gamma$

$\Rightarrow \alpha f = \gamma$, where $f : K/E' \rightarrow E/E'$ is an N-monomorphism and $\phi = Hom_N(f, W)$.

Thus W is $E/E'$-injective.

$\Rightarrow E''$ is $W_cI$ N-group of E.

## 3.3. On direct sum of N-groups with Injectivity and E-injectivity:

In this section we study direct sums of relative injective N-groups and N-subgroups, direct product of relative injective N-groups. Using the notion of dominance of an element of an N-group by another N-group direct sums of relative injective N-groups several properties are established.

**Proposition 3.3.1:** Let N be a dgnr. If $E_\alpha$ is a WI N-group for all $\alpha \in A$ then $E = \oplus_{\alpha \in A} E_\alpha$ is a WI N-group, where E is commutative.

**Proof:** Let $E = \oplus_{\alpha \in A} E_\alpha$ and $E_\alpha$ is WI N-group

$\Rightarrow$ W is $E_\alpha$-injective for all $\alpha \in A$.

We consider an N-subgroup K of E and the N-homomorphism $h : K \to W$.

Let $\Omega = \{ f : L \to W / K \leq L \leq E$ and $(f|K) = h\}$.

Let $g : A \to W, h : B \to W \in \Omega$. $g \leq h$ if $A \subseteq B \subseteq E$.

Then $\Omega$ is ordered set by set inclusion. $\Omega$ is clearly inductive.

Let $\bar{h} : M \to W$ be a maximal element in $\Omega$.

To get the proof it is sufficient to show that each $E_\alpha$ is contained in M.

Let $K_\alpha = E_\alpha \cap M$.

Then $(\bar{h} \mid K_\alpha)$: $K_\alpha \to W$, so since $K_\alpha \leq E_\alpha$ and W is $E_\alpha$- injective, there is an N-homomorphism

$\overline{h_\alpha} : E_\alpha \to W$ with $(\overline{h_\alpha} \mid K_\alpha) = (\bar{h} \mid K_\alpha)$.

If $e_\alpha \in E_\alpha$ and $m \in M$ such that $e_\alpha + m = 0$, then $e_\alpha = -m \in K_\alpha$ and $\overline{h_\alpha}(e_\alpha) + \bar{h}(m)$

$= \bar{h}(-m) + \bar{h}(m) = 0.$

Thus $f : e_\alpha + m \to \overline{h_\alpha}(e_\alpha) + \bar{h}(m)$ is a well defined N-homomorphism $f : E_\alpha + M \to W$.

But $(f|M) = \bar{h}$, so by maximality of $\bar{h}$, $E_\alpha \subseteq M$.

**Proposition 3.3.2**: W is E- injective $\Rightarrow$ W is Ne–injective for all $e \in E$.

**Proof:** Since Ne is an N-subgroup of E. As W is E- injective, proposition 3.2.15 implies W is Ne-injective.

**Proposition 3.3.3**: Let N be a dgnr. If W is a commutative N-group and $\{Ne\}_{e\in E}$ is an independent family of normal N-subgroups of N-group E, W is Ne–injective for all $e \in E$, then W is E- injective

**Proof:** W is Ne–injective for all $e \in E$.

So by proposition 3.3.1, W is $\oplus_{e\in E} Ne$ –injective.

Since E is a homomorphic image of $\oplus_{e\in E} Ne$ by proposition 3.1.8 and since homomorphic image of a $W_C I$ N-group is $W_C I$ N-group by proposition 3.2.16.

So W is E-injective.

**Proposition 3.3.4:** If a finite direct sum of injective normal N-subgroups (ideals) of E, i.e.

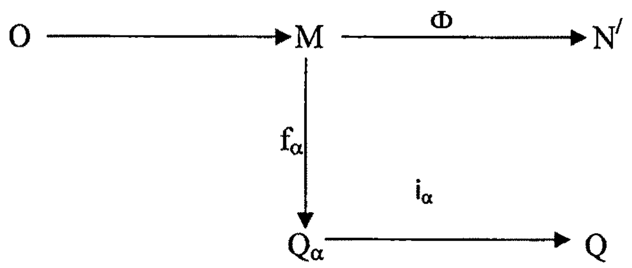
$Q = \oplus Q_\alpha$, where $Q_\alpha$ is normal N-subgroup (or ideal) of E is injective, then each $Q_\alpha$ is injective .

**Proof:** Let $Q = \oplus Q_\alpha$ be injective N- subgroup and consider the N-monomorphism

$f_\alpha : M \to Q_\alpha$, where M is some N- subgroup of E.

∵ Q is direct sum, for any $\alpha = 1, 2, 3, \ldots\ldots\ldots, n$ there is the inclusion map $i_\alpha : Q_\alpha \to Q$ and the projection on $\Pi_\alpha : Q \to Q_\alpha$ such that $\Pi_\alpha i_\alpha = 1_{Q_\alpha}$.

Consider a diagram



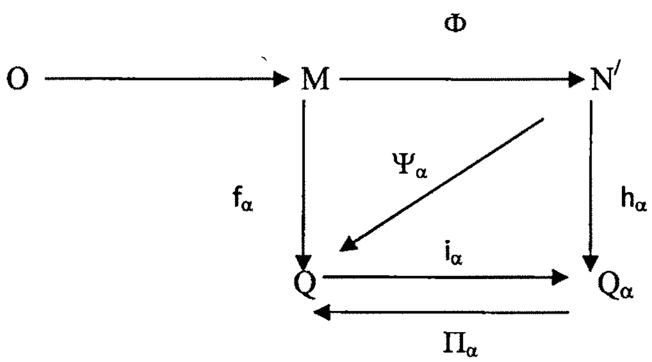

with top row exact.

Since Q is injective ∃ an N- homomorphism $h_\alpha : N' \to Q$, such that $h_\alpha \Phi = i_\alpha f_\alpha$.

Now define $\Psi : N' \to Q_\alpha$ by $\Psi_\alpha = \Pi_\alpha h_\alpha$.

Since $\Pi_\alpha i_\alpha = 1_{Q_\alpha}$, it follows that $\Psi_\alpha \Phi = \Pi_\alpha h_\alpha \Phi = \Pi_\alpha i_\alpha f_\alpha = f_\alpha$.

So, the diagram




commutative.

Thus $Q_\alpha$ is injective.

**Proposition 3.3.5:** Let N be a dgnr. A finite direct sum of injective normal N-subgroups (ideals) of E, i.e. $Q = \oplus Q_\alpha$, where $Q_\alpha$ is normal N-subgroup (or ideal) of E, is injective if each $Q_\alpha$ is injective .

**Proof:** Let $Q = \oplus Q_\alpha$ with each $Q_\alpha$ injective N- group.

Now consider a diagram

$$\begin{array}{ccccc} O & \longrightarrow & M & \xrightarrow{\Phi} & N' \\ & & \downarrow f & & \\ & & Q, & & \end{array}$$

where M, $N'$ are N subgroups of E with the top row exact.

For any $\alpha = 1, 2, 3, \ldots \ldots \ldots, n$, there is the canonical inclusion $i_\alpha : Q_\alpha \to Q$ and the projection $\Pi_\alpha : Q \to Q_\alpha$, so there are the N-homomorphisms $\Pi_\alpha f : M \to Q_\alpha$.

Since $Q_\alpha$ is injective there exists a N-homomophism $h_\alpha : N' \to Q_\alpha$ such that $h_\alpha \Phi = \Pi_\alpha f$.

Now define a map $h : N' \to Q$ by the formula

$$h(x) = \Sigma \{h_\alpha (x)\}_{\alpha=1}^{n}$$
$$= (h_1(x) + \ldots \ldots \ldots + h_n(x)) \quad \forall x \in N'.$$

Then h is N-homomophism.

Since $h(x_1 + x_2) = (h_1(x_1 + x_2) + \ldots \ldots \ldots + h_n(x_1 + x_2))$

$= (h_1(x_1) + h_1(x_2) + \ldots \ldots \ldots + h_n(x_1) + h_n(x_2))$

$= h_1(x_1) + \ldots \ldots \ldots + h_n(x_1) + h_1(x_2) + \ldots \ldots \ldots + h_n(x_2)$ [ since Q is normal N-subgroup]

$= h(x_1) + h(x_2)$

$h(n'x) = (h_1(n'x) + \ldots \ldots \ldots + h_n(n'x))$

$= h_1(n'x) + \ldots \ldots \ldots + h_n(n'x)$

$= n' h_1(x) + \ldots\ldots\ldots + n'h_n(x)$

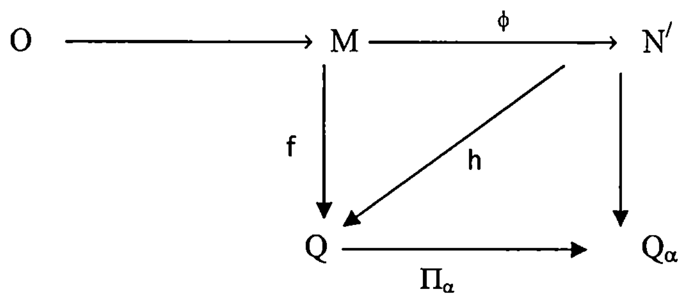
$= \sum_{i=1}^{n} s_i (h_1(x)) + \ldots\ldots\ldots + \sum_{i=1}^{n} s_i (h_n(x))$

$= s_1((h_1(x)) + \ldots\ldots\ldots + h_n(x)) + \ldots\ldots\ldots + s_n((h_1(x)) + \ldots\ldots\ldots + h_n(x))$

$= s_1h(x) + \ldots\ldots\ldots + s_nh(x)$

$=(\sum_{i=1}^{n} s_i)h(x) = n'h(x).$

We shall show the diagram




commutes. i.e. $f = h\Phi$.

Since Q is direct sum, for any $x \in N'$

$h\phi(x) = (h_1\phi(x) + h_2\phi(x) + \ldots\ldots\ldots + h_n \phi(x))$

$= (\Pi_1 f(x) + \Pi_2 f(x) + \ldots\ldots\ldots + \Pi_n f(x))$

$= f(x)$

$\therefore h\phi = f.$

Thus Q is injective.

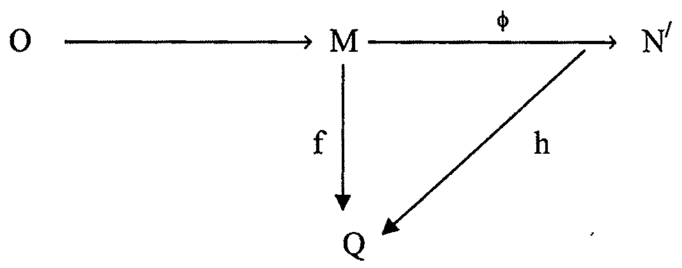
**Corollary 3.3.6:** Let N be a dgnr. A finite direct sum of injective normal N-subgroups (ideals) of E, i.e. $Q = \oplus Q_\alpha$, where $Q_\alpha$ is normal N-subgroup (or ideal) of the group E, is injective if and only if each $Q_\alpha$ is injective .

**Theorem 3.3.7:** A finite direct sum of injective N-groups, that is $Q = \oplus Q_\alpha$, where $Q_\alpha$ is N-groups is injective if and only if each $Q_\alpha$ is injective .

**Proof:** Let Q be injective, to show each $Q_\alpha$ is injective. Proof is same as theorem 3.3.4.

Conversely, let each $Q_\alpha$ be injective, to show Q is injective.

Now consider a diagram




where M, $N'$ are N groups with the top row exact.

For any $\alpha = 1, 2, 3, \ldots \ldots \ldots, n$, there is the canonical inclusion $i_\alpha : Q_\alpha \to Q$ and the projection $\Pi_\alpha : Q \to Q_\alpha$, so there are the N-homomorphisms $\Pi_\alpha f : M \to Q_\alpha$ .

Since $Q_\alpha$ is injective, there exists an N-homomophism $h_\alpha : N' \to Q_\alpha$ such that $h_\alpha \Phi = \Pi_\alpha f$.

Now define a map $h : N' \to Q$ by the formula

$$h(x) = (h_1(x), \ldots \ldots \ldots, h_n(x)) \qquad \forall x \in N'.$$

Then h is N-homomophism.

Since $h(x_1 + x_2) = (h_1(x_1 + x_2), \ldots \ldots \ldots, h_n(x_1 + x_2))$

$$= (h_1(x_1) + h_1(x_2), \ldots \ldots \ldots, h_n(x_1) + h_n(x_2))$$

$$= (h_1(x_1), \ldots \ldots \ldots, h_n(x_1)) + (h_1(x_2), \ldots \ldots \ldots, h_n(x_2))$$

$$= h(x_1) + h(x_2)$$

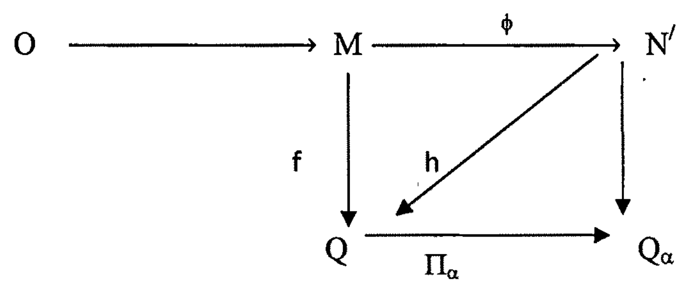
$h(n'x) = (h_1(n'x), \ldots \ldots \ldots, h_n(n'x))$

$= (n' h_1(x), \ldots\ldots\ldots, n' h_n(x))$

$= n' (h_1(x), \ldots\ldots\ldots, h_n(x))$

$= n' h(x).$

We shall show the diagram




commutes. i.e. $f = h\Phi$.

Since Q is direct sum, for any $x \in N'$

$h\phi(x) = (h_1\phi(x), h_2\phi(x), \ldots\ldots\ldots, h_n \phi(x))$

$= (\Pi_1 f(x), \Pi_2 f(x), \ldots\ldots\ldots, \Pi_n f(x))$

$= f(x)$

$\therefore h\phi = f.$

Thus Q is injective.

**Theorem 3.3.8:** Let N be a near-ring and $\{Q_i\}_{i\in I}$ a family of E-injective N-groups. Then the product $Q = \Pi_{i\in I} Q_i$ is E- injective.

**Proof:** Let $A \subseteq E$ be an N-subgroup of E and $f : A \to Q$ an N-homomorphism.

It is enough to show f can be extended to E.

For $i \in I$ denote $\pi_i : Q \to Q_i$ the projection map.

Since $Q_i$ is E-injective for any $i \in I$, so the N-homomorphism $\pi_i.f : A \to Q_i$ can be extended to $f_i' : E \to Q_i$. Then we have $f' : E \to Q$ by $f'(e) = (f_i'(e))_{i \in I}$.

If $a \in A$, then $f'(a) = f(a)$, so $f'$ is an extension of f.

Thus Q is E-injective.

**Definition 3.3.9**: For an N-group A an element $x \in A$ is said to be dominated by N-group E if $Ann_N(x) \supset Ann_N(e)$ for some $e \in E$.

Given a family $\{A_\alpha\}_{\alpha \in J}$ of N-groups. Let x be the element of $\Pi_{\alpha \in J} A_\alpha$ whose $\alpha$-component is $x_\alpha$.

We define $I_x = \{n \in N / nx \in \oplus_{\alpha \in J} A_\alpha\}$.

Then $x \in \Pi_{\alpha \in J} A_\alpha$ is called a special element if $I_x x_\alpha = 0$ for almost all $\alpha$. In other words $\exists$ a finite subset F of J such that $nx_\alpha = 0$ for all $n \in I_x$ and for all $\alpha \in F$.

**Theorem 3.3.10:** If $\oplus_{\alpha \in J} A_\alpha$ is E -injective then each $A_\alpha$ is E –injective and every element of $\Pi_{\alpha \in J} A_\alpha$ dominated by E is special.

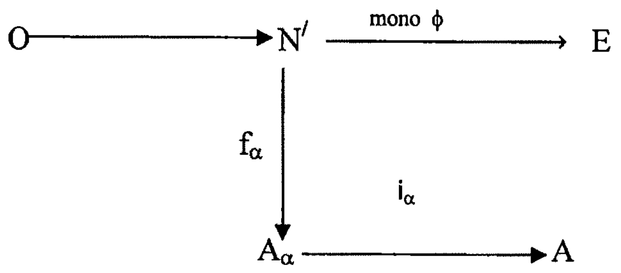
**Proof:** Let $A = \oplus_{\alpha \in J} A_\alpha$ be E injective.

Consider the N-homomorphism $f_\alpha : N' \to A_\alpha$.

$\because$ A is direct sum, $N'$ some N-group of N for any $\alpha \in J$, there is the inclusion map

$i_\alpha : A_\alpha \to A$ and the projection $\pi_\alpha : A \to A_\alpha$ such that $\pi_\alpha i_\alpha = 1_{A_\alpha}$.

Consider a diagram,



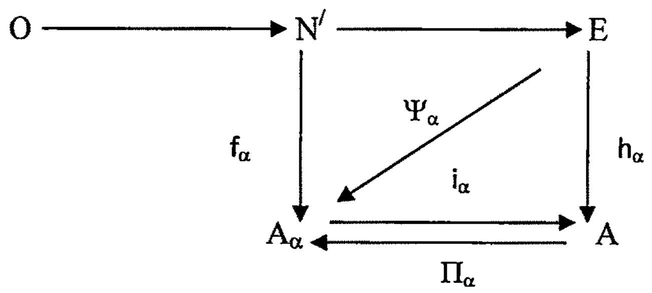

with top row exact.

Since A is E - injective, $\exists$ a homomorphism $h_\alpha : E \rightarrow A$ such that $h_\alpha \Phi = i_\alpha f_\alpha$.

Now define $\Psi_\alpha : E \rightarrow A_\alpha$ by $\Psi_\alpha = \pi_\alpha h_\alpha$.

Since $\pi_\alpha i_\alpha = 1_{A_\alpha}$, it follows that $\Psi_\alpha \Phi = \pi_\alpha h_\alpha \Phi = \pi_\alpha i_\alpha f_\alpha = f_\alpha$

So the diagram




commutative.

Thus $A_\alpha$ is E -injective.

Let $x \in \Pi_\alpha A_\alpha$ be dominated by E $\Rightarrow$ there is an $e \in E$ such that $Ann_N(x) \supset Ann_N(e)$.

Then it gives an N-homomorphism $f : Ne \rightarrow \Pi A_\alpha$ defined by $\lambda e \rightarrow \lambda x$ $(\lambda \in N)$.

Let $(\lambda_1 e), (\lambda_2 e) \in Ne$ and

$f(\lambda_1 e) \neq f(\lambda_2 e)$

$\Rightarrow (\lambda_1 x) \neq (\lambda_2 x)$

$\Rightarrow (\lambda_1 - \lambda_2)x \neq 0$

$\Rightarrow (\lambda_1 - \lambda_2) \notin Ann_N(x)$

$\Rightarrow (\lambda_1 - \lambda_2) \notin Ann_N(e)$ [since $Ann_N(x) \supset Ann_N(e)$]

$\Rightarrow (\lambda_1 - \lambda_2)e \neq 0$

$\Rightarrow (\lambda_1 e) \neq (\lambda_2 e)$

$\therefore$ the mapping is well defined .

$$f(\lambda_1 e + \lambda_2 e) = f((\lambda_1 + \lambda_2)e)$$

$$= (\lambda_1 + \lambda_2)x$$

$$= (\lambda_1 x + \lambda_2 x)$$

$$= f(\lambda_1 e) + f(\lambda_2 e)$$

Next for $n \in N$, $f(n(\lambda_1 e)) = f((n\lambda_1)e)$

$$= (n\lambda_1)x$$

$$= n(\lambda_1 x)$$

$$= n\, f(\lambda_1 e)$$

Thus f is an N-homomorphism .

The image of the N-subgroup $I_x e$ by f is clearly $I_x x$ ( $\subset \oplus A_\alpha$ ).

Thus the restriction of f to $I_x e$ is regarded as an N-homomorphism $I_x e \rightarrow \oplus A_\alpha$.

Since $\oplus A_\alpha$ is E-injective and so Ne-injective by proposition 3.3.2.

So, we get N-homomorphism $Ne \rightarrow \oplus A_\alpha$ which means that there exists a $u \in \oplus A_\alpha$ such that $\lambda x = \lambda u$ (for all $\lambda \in I_x$).

It follows that $I_x x_\alpha = I_x u_\alpha$ for all $\alpha \in J$.

But since $u_\alpha = 0$ for almost all $\alpha$, it follows that $I_x x_\alpha = 0$ for almost all $\alpha$ too

$\Rightarrow$ x is special.

**Theorem 3.3.11:** If $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of N-group E in a dgnr near-ring N , $\oplus_{\alpha \in J} A_\alpha$ is commutative N-group then each $A_\alpha$ is E –injective and every element of $\Pi_{\alpha \in J} A_\alpha$ dominated by E is special implies $\oplus_{\alpha \in J} A_\alpha$ is E -injective .

**Proof:** let each $A_\alpha$ is E -injective and every element of $\Pi_{\alpha \in J} A_\alpha$ dominated by E is special.

Let $e \in E$ and consider the N- subgroup Ne of E.

Let J be an N-subgroup of N.

Then Je is an N-subgroup of Ne.

[Let se, te $\in$ Je, s, t $\in$ J, se + te = (s + t)e $\in$ Je and for n $\in$ N, n(se) = (ns)e $\in$ Je, since ns $\in$ J as J is N-subgroup of N]

Let there be given an N- homomorphism h : Je $\rightarrow \oplus A_\alpha$.

Then since $\oplus A_\alpha \subset \Pi A_\alpha$ and $\Pi A_\alpha$ is E-injective (as each $A_\alpha$ is E-injective, by proposition 3.3.8) whence Ne- injective (by proposition 3.3.2), h can be extended to an N-homomorphism Ne $\rightarrow \Pi A_\alpha$.

Let $x \in \Pi A_\alpha$ and we define the N-homomorphism as $\lambda e \rightarrow \lambda x$ $(\lambda \in N)$

Therefore it follows that Jx = h (Je) $\subset \oplus A_\alpha$, whence $J \subset I_x$.

On the otherhand since clearly $Ann_N(e) \subset Ann_N(x)$, x is dominated by E and thus x is special by assumption

$\Rightarrow I_x x_\alpha = 0$ whence $Jx_\alpha = 0$ for almost all $\alpha$.

Let u be the element of $\oplus A_\alpha$, whose $\alpha$-component is $x_\alpha$ or 0 according as $Jx_\alpha \neq 0$ or $Jx_\alpha = 0$.

Then it is clear that $\lambda u = \lambda x$ for all $\lambda \in J$.

Further, it is also clear that $Ann_N(e) \subset Ann_N(x) \subset J$ and therefore the mapping gives an N-homomorphism $f : Ne \rightarrow \oplus A_\alpha$ which is an extension of h, because $f(\lambda e) = \lambda u = \lambda x \ \forall \lambda \in J$.

This implies that $\oplus A_\alpha$ is Ne-injective and so E–injective by proposition 3.3.3.

**Corollary 3.3.12:** Let N be a dgnr. If $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of N-group E, $\oplus_{\alpha \in J} A_\alpha$ is commutative N-group then $\oplus_{\alpha \in J} A_\alpha$ is E -injective if and only if each $A_\alpha$ is E-injective and every element of $\Pi_{\alpha \in J} A_\alpha$ dominated by E is special implies $\oplus_{\alpha \in J} A_\alpha$ is E-injective

**Theorem 3.3.13:** Suppose $\{ A_\alpha \}_{\alpha \in J}$ is a family of E-injective N-groups such that for every countable subset k of J, $\oplus_{\alpha \in K} A_\alpha$ is E - injective. Then $\oplus_{\alpha \in J} A_\alpha$ is itself E - injective.

**Proof:** Assume that $\oplus_{\alpha \in J} A_\alpha$ is not E - injective.

Then by theorem 3.3.10, there exists an $x \in \Pi_{\alpha \in J} A_\alpha$ which is dominated by E but is not special $\Rightarrow I_x x_\alpha \neq 0$ for infinitely many $\alpha \in J$.

Let k be an infinite countable subset of the infinite set $\{\alpha \in J \,/\, I_x x_\alpha \neq 0\}$.

Let y be element of $\Pi_{\alpha \in k} A_\alpha$, whose $\alpha$- component $y_\alpha$ is equal to $x_\alpha$ for all $\alpha \in K$.

Then clearly $I_x \subset I_y$, so that it follows that y is dominated by E and $I_y y_\alpha = I_y x_\alpha \neq 0 \ \forall \alpha \in K$.

This implies again by theorem 3.3.10, that $\oplus_{\alpha \in K} A_\alpha$ is not E -injective (because each $A_\alpha$ is E - injective by our assumption). This is a contradiction and so the proof is complete.

## 3.4: E-injective and injective N-groups with chain conditions:

In this section we study E-injective N-groups with chain conditions. In particular, E-injective N-groups with descending chain condition are investigated. It is shown that the singular and semi-simple characters play a vital role in characterization of E-injective N-groups.

**Theorem 3.4.1:** Let N be dgnr. If $\{Ne\}_{e\in E}$ is an independent family of normal N-subgroups of N-group E, $\oplus_{\alpha\in J} A_\alpha$ is commutative N-group then direct sum of any family $\{ A_\alpha\}$ of E - injective N- groups is E - injective if E is Noetherian.

**Proof:** let $\{ A_\alpha\}$ be a family of E-injective N- group.

Let x be an element of $\Pi A_\alpha$, dominated by e.

Then there is an $e \in E$ such that $Ann_N(e) \subset Ann_N(x)$.

Consider $I_x e$ .

Since clearly $Ann_N(x) \subset I_x$, whence $Ann_N(e) \subset I_x$, it follows that $I_x/ Ann_N(e) \cong I_x e$.

On the other hand $I_x e$ is a N-subgroup of Ne, so N subgroup of Noetherian N-group E.

Hence, $I_x/ Ann_N(e)$ is finitely generated

$\Rightarrow$ there exists a finite number of elements $\lambda_1, \lambda_2, \dots \dots \dots, \lambda_n$ of $I_x$ such that

$I_x = N\lambda_1 + N \lambda_2 + \dots \dots \dots + N\lambda_n + Ann_N(e)$

It follows therefore

$I_x x_\alpha = N\lambda_1 x_\alpha + N \lambda_2 x_\alpha + \dots \dots \dots + N\lambda_n x_\alpha$ for all components $x_\alpha$.

Since however for each i, $\lambda_i x_\alpha = 0$, for almost all $\alpha$, it follows that $I_x x_\alpha = 0$ for almost all $\alpha$

$\Rightarrow$ x is special.

Thus $\oplus$ $A_\alpha$ is E- injective by theorem 3.3.11.

**Proposition 3.4.2:** If $\{Ne\}_{e\in E}$ is an independent family of normal N-subgroups of N-group E in a dgnr near-ring N , direct sum of E-injective N-groups is commutative N-group then E is Noetherian V N-group($V_c$ N-group) implies every strictly semi- simple N-group is E-injective.

**Proof:** E is Noetherian V- N-group

$\Rightarrow$ E is Noetherian and every simple N-group is E- injective.

Again direct sum of E-injective N-groups is E- injective as E is Noetherian

(by theorem 3.4.1).

Let K be any strictly semi simple N-group

$\Rightarrow$ K is direct sum of simple normal N-subgroups.

So K is E- injective.

**Proposition 3.4.3:** For a finitely generated N-group E every countably generated strictly semi- simple N-group is E- injective implies E is weakly Noetherian $V_c$ N-group.

**Proof:** Suppose $\{A_\alpha\}_{\alpha\in J}$ is a family of N-groups such that for every countable subset K of J, $\oplus_{\alpha\in K} A_\alpha$ is E- injective. Then by theorem 3.3.13 $\oplus_{\alpha\in J} A_\alpha$ itself E-injective.

Now given that every countably generated strictly semi simple N-group is E-injective.

To show E is weakly Noetherian and every simple commutative N-group is E-injective.

Let U be a countably generated strictly semi- simple N-group.

Then $U = \oplus U_\alpha$, where $U_\alpha$ is simple normal N-subgroups, so $U_\alpha$'s can be taken as commutative N-groups and $\alpha \in K$, K is countable subset of J (as U countably generated).

Given U is E-injective. So we have $\oplus U_\alpha$, $\alpha \in J$ is also E-injective (By theorem 3.3.13).

So by theorem 3.3.10, we get every $U_\alpha$ is E-injective

$\Rightarrow$ E is $V_c$ N-group.

Next to show E is weakly Noetherian.

Given E is finitely generated and W countably generated semi-simple N-group & W is E-injective.

Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots\ldots\ldots$ be an ascending chain of distinct ideals of E.

Let $f_K : N_K \to W$ $\quad (k = 1, 2, 3, \ldots\ldots\ldots \infty)$

As W is E-injective, for inclusion map $i_K : N_K \to E$, $\exists$ a map $\gamma_K : E \to W$ s.t. $f_K = \gamma_K\, i_K$

Let $N' = \sum_{k=1}^{\infty} N_k$

Define the map $f : N' \to W$ by

$f(x) = \sum_{k=1}^{\infty} f_k(x)$

$= \sum_{k=1}^{\infty} \gamma_k i_k(x)$

f is well defined.

$\because$ W is E-injective, $\exists$ a map $g : E \to W$ extending f.

But E is finitely generated & g (E) $\subseteq$ W, w countably generated. So g can be defined as

$g(x) = \sum_{k=1}^{m} \gamma_k i_k(x)$

for some positive integer m, which gives chain of ideals must be finite.

**Corollary 3.4.4:** For a finitely generated N-group E, every strictly semi- simple N-group is E- injective implies E is weakly Noetherian $V_c$ N-group.

**Proposition 3.4.5**: For dgnr N, if E is a finitely generated $S^3$I-N-group, then $\frac{E}{Soc(E)}$ is a weakly Noetherian $V_c$ N-group.

**Proof**: From the above corollary 3.4.4, it is enough to show that every strictly semi-simple N-group is $\frac{E}{Soc(E)}$ injective.

Let L be a strictly semi-simple N-group

So as N dgnr, L is a semi-simple N-group.

$\frac{M}{Soc(E)}$ an ideal of $\frac{E}{Soc(E)}$. $f: \frac{M}{Soc(E)} \longrightarrow L$ is a non-zero N-homomorphism.

Let $\frac{K}{Soc(E)} = Kerf$.

We claim K is essential ideal in M.

For if $K \cap I = 0$ for some non-zero ideal I of M then $I \cong \frac{I+K}{K}$ and since the latter is isomorphic to an ideal of L, it follows that for some ideal $I_1 \neq 0$ and contained in I that $I_1 \subset L$, hence $I_1 \subseteq Soc(E) \subseteq K$, a contradiction.

Now $M/_K$ singuler, we may take L singular, since $f(M/_K) \subseteq Z(L)$.

Let $\eta : M \longrightarrow \frac{M}{Soc(E)}$ denote the quotient map and consider the map $f.\eta : M \longrightarrow L$.

$\because$ L is E-injective $f.\eta$ extends to a map of E into L.

$\because Soc(E) \subseteq K$. This yields a map of $\frac{E}{Soc(E)}$ into L by proposition 3.2.3.

**Proposition 3.4.6**: Let N be a dgnr If E is an N-group satisfying the following conditions

(i) $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of E,

(ii) direct sum of E-injective N-groups is a commutative N-group

(iii) No non-zero homomorphic image of Nx, $\forall x(\neq 0) \in Soc(E)$, is semi-simple, singular

(iv) $\frac{E}{Soc(E)}$ is Noetherian V N-group,

then E is an $S^3I$-N-group.

**Proof:** Let L be a strictly semi-simple singular N-group.

Let M be an N-subgroup of E.

$f : M \to L$ a non-zero map with kerf = K.

Then by given condition $Soc(E) \cap M$ is contained in K.

[For $x \in Soc(E) \cap M \Rightarrow x \in Soc(E), x \in M \Rightarrow Nx \subseteq Soc(E), Nx \subseteq M \Rightarrow Nx \in Soc(E) \cap M$].

So by proposition 3.2.3, $\exists$ an N-homomorphism $f' : \frac{M}{Soc(E) \cap M} \to L$.

Since $\frac{M}{Soc(E) \cap M} \cong \frac{Soc(E)+M}{Soc(E)}$, so $f' : \frac{Soc(E)+M}{Soc(E)} \to L$.

As $\frac{E}{Soc(E)}$ is Noetherian V N-group and L semi-simple singular by proposition 3.4.2, L is $\frac{E}{Soc(E)}$ –injective, that is $f'$ is extended to $g' : \frac{E}{Soc(E)} \to L$.

If we define $g : E \to L$ by $g(e) = g'(\bar{e} + Soc(E))$. g is extension of f.

**Proposition 3.4.7:** Let E be an N-group. Then $E/_M$ is weakly Noetherian for every essential ideal M of E if and only if E has A.C.C. on essential ideals.

**Proof:** Let M be an essential ideal of E .

Then E/M weakly Noetherian

We show E has A.C.C. on essential ideals.

Let $M_1 \subset M_2 \subset M_3 \subset \ldots \ldots \ldots \rightarrow$ (1) be a chain of ideals of E where $M_i \leq_e E$.

Considering an essential N-subgroup $M \subseteq M_i$ $\forall$ i, we can construct another chain

$M_1/M \subset M_2/M \subset M_3/M \subset \ldots \ldots \ldots$ of E/M.

Since E/M is weakly Noetherian we get $M_i/M = M_{i+1}/M$ for some i.

Now $M_i \subset M_{i+1}$. Our aim is to show $M_{i+1} \subset M_i$.

Let $x_{i+1} \in M_{i+1}$ but $x_{i+1} \notin M$.

Then $x_{i+1} + M \in M_{i+1}/M \Rightarrow x_{i+1} + M \in M_i/M \Rightarrow x_{i+1} \in M_i$ ( since $x_{i+1} \notin M$).

So $M_i = M_{i+1}$.

$\Rightarrow$ E has A.C.C. on essential ideals.

Converse is clear.

**Proposition 3.4.8**: N-group E is almost weakly Noetherian if and only if $E/_M$ is weakly Noetherian for every essential ideal M of E.

**Proof:** Let $E/_{SocE}$ be weakly Noetherian.

We know if N ideal of M, M weakly Noetherian $\Leftrightarrow$ N & $M/_N$ weakly Noetherian, by proposition 4.1.7.

M is essential ideal of E and SocE is the intersection of all essential ideals $\Rightarrow$ Soc E $\subseteq$ M.

$\Rightarrow {}^{E}/_{SocE}$ is weakly Noetherian $\Leftrightarrow {}^{M}/_{SocE}$ and $\frac{E/_{SocE}}{M/_{SocE}} \cong {}^{E}/_{M}$ weakly Noetherian.

Conversely, ${}^{E}/_{M}$ is weakly Noetherian for every essential ideal M of E.

We show $\frac{E}{SocE}$ is weakly Noetherian. It is enough to show that every essential ideal of $\frac{E}{SocE}$ is finitely generated by proposition 3.4.7.

Let $\frac{M}{SocE}$ be an essential ideal of $\frac{E}{SocE}$.

Let k be an ideal of M maximal with respect to K $\cap$ SocE = 0.

Then K $\oplus$ SocE is essential in M and hence essential in E.

[K $\oplus$ SocE ideal of M. let $M'$ ideal of M such that $M' \cap$ ( K $\oplus$ SocE ) = 0.Then $M' \oplus$( K $\oplus$ SocE ) is a direct sum $\Rightarrow$ $M' \oplus$ K $\oplus$ SocE is a direct sum. Whence ( $M' \oplus$ K) $\cap$ SocE $=$ 0. By maximalilty of K, ( $M' \oplus$ K) = K, i.e $M' = 0$.]

Then $\frac{E}{K \oplus SocE}$ is weakly Noetherian. So $\frac{M}{K \oplus SocE}$ is finitely generated.

From the exactness of the sequence $0 \to K \to \frac{M}{SocE} \to \frac{M}{K \oplus SocE} \to 0$, it suffices to show K is finitely generated.

We claim that K is finite dimensional.

For, if not $\exists$ an infinite direct sum of non-zero ideals $\oplus_{i \in I} K_i$ which is essential in K.

Since $K_i \cap$ Soc E = 0, each $K_i$ has a proper essential ideal $T_i$.

[since $K_i \cap$ SocE = Soc $K_i$ = 0].

Let $T = \oplus_{i\in I} T_i$.

Then T is an essential ideal of K.

Let $K'$ be an ideal of K, $T = \oplus_{i\in I} T_i$, where $T_i$ are essential ideals of $K_i$.

Now $K' = \oplus_{i\in I} K_i'$, $K_i' \subseteq K_i$. Then $T_i \cap K_i' \neq 0$

$\Rightarrow \oplus_{i\in I} T_i \cap K_i' \neq 0$

$\Rightarrow T \cap \oplus_{i\in I} K_i' \neq 0.$

$\Rightarrow T \cap K' \neq 0.$

Again SocE is an essential ideal of SocE and $T \cap \text{SocE} = 0$.

So $T \oplus \text{SocE} \leq_e K \oplus \text{SocE} \Rightarrow T \oplus \text{SocE}$ is an essential ideal of E.

Hence $E/_{T \oplus \text{SocE}}$ is weakly Noetherian,

As ideal of a weakly Noetherian N-group is weakly Noetherian, $\frac{\oplus_{i\in I} K_i}{T \oplus \text{SocE}}$ is weakly Noetherian

$\Rightarrow \frac{\oplus_{i\in I} T_i}{T \oplus \text{SocE}}$ is weakly Noetherian.

$\frac{\oplus_{i\in I} T_i}{T \oplus \text{SocE}} \subseteq \frac{\oplus_{i\in I} K_i}{T \oplus \text{SocE}}$ and $\frac{\oplus_{i\in I} K_i}{T \oplus \text{SocE}}$ weakly Noetherian imply $\frac{\frac{\oplus_{i\in I} K_i}{T \oplus \text{SocE}}}{\frac{\oplus_{i\in I} T_i}{T \oplus \text{SocE}}} \cong \frac{\oplus_{i\in I} K_i}{\oplus_{i\in I} T_i} \cong \oplus_{i\in I} \frac{K_i}{T_i}$ is weakly Noetherian, a contradiction, since it is an infinite direct sum of non zero N-groups.

Thus K is finite dimensional.

Let $(K_i)_{i=1}^{n}$ be a family of non-zero ideals of K such that $\oplus_{i=1}^{n} K_i$ is essential in K.

$\Rightarrow \oplus_{i=1}^{n} K_i \leq_e K$, so $\oplus_{i=1}^{n} K_i \oplus \text{SocE} \leq_e K \oplus \text{SocE} \leq_e E$.

$\Rightarrow \oplus_{i=1}^{n} K_i \oplus SocE \leq_e E$.

$\Rightarrow \frac{E}{\oplus_{i=1}^{n} K_i \oplus Soc\, E}$ is weakly Noetherian.

We define $f: \frac{K}{\oplus_{i=1}^{n} K_i} \to \frac{K}{\oplus_{i=1}^{n} K_i \oplus Soc\, E}$ by $f(k + \oplus_{i=1}^{n} K_i) = f(k + \oplus_{i=1}^{n} K_i \oplus SocE)$

Now $f(k_1 + \oplus_{i=1}^{n} K_1) \neq f(k_2 + \oplus_{i=1}^{n} K_i)$

$\Rightarrow (k_1 + \oplus_{i=1}^{n} K_i \oplus SocE) \neq (k_2 + \oplus_{i=1}^{n} K_i \oplus SocE)$

Next, let $\bar{k} \in \frac{K}{\oplus_{i=1}^{n} K_i \oplus Soc\, E}$.

If $\bar{k} = k_1 + \oplus_{i=1}^{n} K_i \oplus Soc\, E$, $\exists\, k_1 + (\oplus_{i=1}^{n} K_i) \in \frac{K}{\oplus_{i=1}^{n} K_i}$ such that

$f(k_1 + (\oplus_{i=1}^{n} K_i)) = k_1 + (\oplus_{i=1}^{n} K_i \oplus L)$.

So f is onto, that is f is isomorphism.

Thus $\frac{K}{\oplus_{i=1}^{n} K_i}$ is isomorphic to the ideal $\frac{K}{\oplus_{i=1}^{n} K_i \oplus Soc\, E}$ of weakly noetherian N-group $\frac{E}{\oplus_{i=1}^{n} K_i \oplus Soc\, E}$ .So we have that $\frac{K}{\oplus_{i=1}^{n} K_i}$ is finitely generated, whence K is finitely generated.

Thus $\frac{E}{SocE}$ is weakly Noetherian.

**Proposition 3.4.9**: If N-group E is almost weakly Noetherian then E has A.C.C. on essential ideals.

**Proof:** Given $\frac{E}{SocE}$ is weakly Noetherian.

To show E has A.C.C. on essential ideals.

Soc E is the intersection of all essential ideals of E.

Hence if $\frac{E}{SocE}$ is weakly Noetherian, E has A.C.C. on essential ideals.

**Proposition 3.4.10**: Let N be a dgnr. If N-group E has A.C.C. on essential ideals then E is almost weakly Noetherian.

**Proof:** We assume that E has A.C.C. on essential ideals.

Let $A \subseteq B$ be ideals of M such that A is essential in B.

By Zorn's lemma there is a maximal ideal L of E such that $L \cap A = 0$ .

And $A \oplus L$ is essential in E.

Since $A + L = A \oplus L$, so that $A \oplus L$ is an ideal of E. Let C ideal of E with $C \cap (A \oplus L)$ $=0$. Then $(A \oplus L) \oplus) C$ is direct $\Rightarrow (A \oplus L) + C = (A \oplus L \oplus C)$ whence $A \cap (L \oplus C) =$ $0$. By maximality of L we obtain $L \oplus C = L$ Thus $C = 0$. $\therefore$ $A \oplus L$ essential ideal of E.

Hence $E/ (A \oplus L)$ satisfies ACC on its ideals.

We consider the map $\phi : B \oplus L \longrightarrow B/A$ by $b + l \longrightarrow b + A$. [N dgnr]

Now $\phi(b_1 + l_1 + b_2 + l_2)$

$= \phi( b_1 + b_2 + l_1 + l_2)$

$=( b_1 + b_2 ) + A$

$= b_1 + A + b_2 + A$

$=\phi(b_1 + l_1) + \phi (b_2 + l_2)$

Again, $\phi n(b + l)$

$= \phi(n_1 + n_2 + n_3 + \dots + n_k)(b + l)$

$= \phi\{ n_1(b + l) + n_2(b + l) + \ldots + n_k(b + l)\}$

$= \phi\{ (n_1b + n_1l) + (n_2b + n_2l) + \ldots + (n_kb + n_kl)\}$

$= (n_1b + A) + (n_2b + A) + \ldots + (n_kb + A)$

$= (n_1b + n_2b + \ldots + n_kb) + A$

$= nb + A$

$= n(b + A)$

$= n\phi(b + l)$

So $\phi$ is an N-homomorphism.

$\text{Ker}\phi = \{ \bar{x} / \phi(\bar{x}) = A\}$

$= \{ a + l / \phi(a + l) = A\}$

$= A + L$

As $A \leq B$ and $B \cap L = 0$, $A \cap L = 0$.

$\therefore$ $\text{Ker}\phi = A \oplus L$

So $B/A \cong (B \oplus L)/(A \oplus L)$.

Hence we get B/A also satisfies acc on its ideals.

In particular, every uniform ideal of E satisfies acc on its ideals.

Since if I is uniform ideal of E and $J_1 \subseteq J_2 \subseteq \ldots \ldots \ldots$ an ascending chain of ideals of I. As I is uniform, each $J_i \leq_e I$.

$\Rightarrow I/J_i$ satisfies acc on its ideals.

$\Rightarrow$ I satisfies acc on essential ideals.(by proposition 3.4.7)

As each $J_i \leq_e I$, $\exists t$ such that $J_t = J_{t+1} \Rightarrow$ I satisfies acc on its ideals.

Now, let H be an ideal of E which is maximal with respect to the condition $H \cap Soc(E) = 0$.

Then $H \oplus Soc(E)$ is essential in E and $E/_{H \oplus Soc(E)}$ satisfies acc on its ideals.

Hence for proving that $E/Soc(E)$ satisfies acc on its ideals it is enough to prove that H satisfies acc on its ideals.

We first show that H has finite Goldie dimension.

Assume that H contains an infinite direct sum $X = X_1 \oplus X_2 \oplus \ldots \ldots \ldots$ of non-zero ideals $X_i$.

Since, $Soc(X_i) = X_i \cap Soc(E)$, each $X_i$ contains a proper essential ideal $Y_i$ and

$Y = Y_1 \oplus Y_2 \oplus \ldots \ldots \ldots$ is an essential ideal of X.

By the above $X/_Y$ satisfies acc on its ideals.

But this is impossible because

$X/_Y = X_1/_{Y_1} \oplus X_2/_{Y_2} \oplus \ldots \ldots \ldots$ with each $X_i/_{Y_i}$ non zero.

This contradiction shows that H has finite Goldie dimension k (say). Then H contains k independent uniform ideals $U_i$ such that $U = U_1 \oplus U_2 \oplus \ldots \ldots \ldots \oplus U_k$ is essential in H.

By the above U and H/U satisfies acc on ideals.

Hence H satisfies acc on ideals.

**Proposition 3.4.11**: if E is non-singular and Every singular homomorphic image of E is weakly Noetherian then E is almost weakly Noetherian.

**Proof:** As M is essential ideal of E and E is non-singular, $E/_M$ is singular.

Again $E/_M$ is homomorphic image of E, by given condition $E/_M$ is weakly Noetherian.

**Proposition 3.4.12**: E is non-singular and almost weakly Noetherian and in E every weakly essential N-subgroup is essential then every singular homomorphic image of E is weakly Noetherian.

**Proof:** Let $f : E \longrightarrow L$ be an N-epimorphism and L is singular.

Now E is non-singular and $\ker f \subseteq E$, $L \cong E/_{\ker f}$ singular,

so $\ker f \leq_{we} E$ by proposition 3.1.7.

Then $Soc(E) \subseteq \ker f$.

So by proposition 3.2.3 we get $L \cong E/_{Soc(E)}$.

As E is almost weakly Noetherian, L is weakly Noetherian.

**Corollary 3.4.13:** The following conditions on an N-group E of a dgnr near-ring N are equivalent:

i. E is almost weakly Noetherian.

ii. $E/_M$ is weakly Noetherian for every essential ideal M of E..

iii. E has A.C.C. on essential ideals.

Moreover if E is non-singular, every weakly essential N-subgroup is essential then above conditions are equivalent to

iv. Every singular homomorphic image of E is weakly Noetherian.

**Proposition 3.4.14**: Near-ring N is weakly Noetherian if $\oplus_{i\in I} E_i$ of injective N-groups is injective.

**Proof:** Let $\oplus_{i \in I} E_i$ of commutative N-groups is injective and that

$I_1 \leq I_2 \leq \ldots \ldots \ldots$ be an ascending chain of left ideals in N.

Let $I = \cup_{i=1}^{\infty} I_i$.

If $a \in I$, then $a \in I_i$ for all but finitely many $I \in N$.

So there is an

$f : I \longrightarrow \oplus_{i=1}^{\infty} E(N/I_i)$

defined via $\Pi_i f(a) = a + I_i \quad (a \in I)$.

By theorem 4.1.9, there is an $x \in \oplus_{i=1}^{\infty} E(N/I_i)$ such that $f(a) = ax$ for all $a \in I$. Now choose $n$ such that $\Pi_{n+k} I(X) = 0$, $k = 0, 1, \ldots \ldots \ldots$

So $I/I_{n+k} = \Pi_{n+k}(f(I)) = \Pi_{n+k}(I_x) = I\Pi_{n+k}(x) = 0$

or, equivalently, $I_n = I_{n+k}$ for all $k = 0, 1, 2, \ldots \ldots \ldots$

So, N is weakly Noetherian.

**Definition 3.4.15:** An N-subgroup U of N-group E is called pure in E if $IU = U \cap IE$ for each ideal I of N.

**Example 3.4.16:** $N = \{0, a, b, c\}$ is the Klein's four group with multiplication

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | b | 0 | 0 |
| c | 0 | c | b | c |

Then (N, +, .) is a near-ring. Here A = {0, c} is N-subgroup of $_NN$ and B = {0, b} is ideal of $_NN$.

Now BA = {0} and A $\cap$ BN = {0, c} $\cap$ {0, b} = {0}. So BA = A $\cap$ BN. So, A is pure in $_NN$.

**Proposition 3.4.17:** If N is non-singular, SocN is pure and every injective right N-group is injective as an N/K-group for ideal K of N then direct sum of (countably many) injective hulls of simple weak singular left N-groups is injective implies N is an almost weakly Noetherian near-ring.

**Proof:** Let $\{S_i\}_{i\in I}$ be a family of simple weak singular N/Soc(N)- groups.

Since a simple N-group is weak singular if and only if it is annihilated by Soc(N).

For let E is simple and weak singular. So $Z_w(E) = \{x \in E / Ix = 0, I \leq_{ei} N\} = E$.

So $x \in E \Rightarrow \exists\, I \leq_{ei} N$ such that $Ix = 0 \Rightarrow$ Soc(N) x = 0.Thus E is annihilated by Soc(N).

Again let E is annihilated by Soc(N), we get Soc(N) E = 0.

$\Rightarrow$ Soc(N) $\subseteq$ Ann(E).

Now we show Ann(E) = $\{ x \in N / xE = 0\}$ is essential ideal in N.

If possible Ann(E) is not essential ideal in N.

Then Ann(E) $\cap$ J = 0 for some non-zero ideal J of N.

If $\forall\, x \in E$ $f : J \to Jx$ , defined by f(j) = jx, it is a well defined N-homomorphism.

$f(j_1) \neq f(j_2) \Rightarrow (j_1x) \neq (j_2x) \Rightarrow (j_1 - j_2)x \neq 0 \Rightarrow (j_1 - j_2) \neq 0 \Rightarrow j_1 \neq j_2$. So f is well-defined.

Next let $j_1 \neq j_2 \Rightarrow (j_1 - j_2) \neq 0 \Rightarrow (j_1 - j_2)x \neq 0 \Rightarrow (j_1x) \neq (j_2x) \Rightarrow f(j_1) \neq f(j_2)$.

So f is one-one.

Again for every $jx \in Jx$, $\exists\, j \in J$ such that $f(j) = jx$. So f is onto.

$f(j_1 + j_2) = (j_1 + j_2)x = (j_1 x + j_2 x) = f(j_1) + f(j_2)$,

$f(nj) = (nj)x = n(jx) = nf(j)$. So f is N-isomorphism.

$\Rightarrow \forall\, x \in E,\ J \cong Jx$.

Again $Z(N) = 0 \Rightarrow Z(J) = 0 \Rightarrow Z(Jx) = 0$

$\Rightarrow \forall\, I \leq_{ei} N,\ I(Jx) \neq 0 \Rightarrow SocN.(Jx) \neq 0$.

But $Jx \subseteq E$ and $SocN.E = 0 \Rightarrow SocN.(Jx) = 0$, a contradiction.

So Ann(E) is essential ideal of N, so E is weak singular.

It follows that each ${}_NS_i$ is weak singular as an N- group.

Since SocN is pure we get $Soc({}_NN).E({}_NS_i) \cap {}_NS_i = SocN.S_i,\ \forall\, i \in I$.

As each ${}_NS_i$ is annihilated by Soc(N),

$SocN.S_i = 0$. So $Soc({}_NN).E({}_NS_i) \cap {}_NS_i = 0$. i.e. $\forall\, x \in E({}_NS_i)$, $Soc({}_NN).x \cap {}_NS_i = 0$.

$E({}_NS_i)$ is an essential extension of ${}_NS_i$, and since $Soc({}_NN).x$ is N-subgroup of $E({}_NS_i)$ we get

$\forall\, x \in E({}_NS_i)$, $Soc({}_NN).x = 0$.

Thus $E({}_NS_i)$ is annihilated by Soc(N), $\forall\, i \in I$.

We claim that $\forall\, i \in I$, $E({}_NS_i)$ is weak singular as N-group.

For $x \in E({}_N S_i)$ with $x \notin Z_w(E({}_N S_i))$ then $\forall\, I \leq_{ie} N$, $Ix \neq 0 \Longrightarrow Ann_N(x)$ is not essential in N.

So $Ann_N(x) \cap J = 0$ for some non-zero ideal J of N.

Since $J \cong Jx$ and $Z(N) = 0$, we infer that $Z(Jx) = 0$, whence $Jx \cap S_i = 0$

[Let $Jx \cap S_i \neq 0$ .

$Z(Jx \cap S_i) = 0 \Rightarrow \forall\, I \leq_{ie} N$, $I(Jx \cap S_i) \neq 0 \Rightarrow SocN(Jx \cap S_i) \neq 0$.

But $(Jx \cap S_i) \subseteq E({}_N S_i)$ and $SocN.E({}_N S_i) = 0$,a contradiction].

This implies that $Jx = 0$.

So $J \subseteq Ann_N(x)$, a contradiction.

Now $E({}_{N/Soc(N)} S_i) = \{ x \in E({}_N S_i) : Soc(N)x = 0\} = E({}_N S_i)$ is injective as N-group.

By given condition $\oplus_{i \in I} E_i$ is injective as an N-group and hence injective as N/Soc(N)-group. This implies that N/Soc(N) is weakly Noëtherian by proposition 3.4.14.

*For a distributively generated near-ring we get the following definition, note and three results.*

**Definition 3.4.18 [Pliz]:** The Jacobson-radical of N-group E is the intersection of maximal ideals of E which is maximal as N-subgroup. We denote it by $J_2(E)$

**Note 3.4.19 [Pliz]:** The Jacobson-radical, $J_2(E)$ of N-group E contains all nilpotent N-subgroups of E.

**Lemma 3.4.20:** Let N be a GV- near-ring, then $Z(E) \cap J_2(E) = 0$, for every N-group E.

**Proof:** If $Z(E) = 0$, we are done.

Otherwise let $(0 \neq)\ x \in Z(E)$.

By Zorn's lemma , the set of all ideals M of E with $x \in M$, has a maximal member L.

The quotient N-group $S = (Nx + L)/L$ is simple and singular, therefore E-injective.

[ $Z((Nx + L)/L) = \{ \bar{x} \in (Nx + L)/L \ / \ I\bar{x} = \bar{0}$ for some essential N-subgroup I of N}

Let $\bar{y} \in (Nx + L)/L$ such that $\bar{y} = nx + l + L$.

Now for some essential N-subgroup I in N,

$I\bar{y} = \{n'\bar{y} \ / \ n' \in I\}$

$= \{ (\sum_{i=1}^{k} s_i)(nx + L) \ / \ n' = (\sum_{i=1}^{k} s_i) \in I \}$

$= \{s_1( nx + L ) + s_2( nx + L) + \dots\dots\dots + s_k( nx + L) \ / \ n' \in I \}$

$= \{ s_1nx + L + s_2nx + L + \dots\dots\dots + s_knx + L \ / \ n' \in I \}$

$= \{(s_1nx + s_2nx + \dots\dots\dots + s_knx) + L \ / \ n' \in I \}$ [ since $s_inx \in L$ as $s_in \in N$]

$= \{L\} = \bar{0}$.

So $\bar{y} \in Z(( Nx + L)/L )$]

This means that the natural map of Nx onto S extends to all of E.

The kernel of this extension map is a maximal ideal of E which does not contain x. Whence x can not be in $J_2E$).

So $Z(E) \cap J_2(E) = 0$

**Theorem 3.4.21:** If N is a GV near-ring with A.C.C. on essential ideals and if finite intersection of essential N-subgroups of N is distributively generated, then Z( N) = 0. In particular, if N is $S^3$ I near-ring with unity then it is non-singular.

**Proof:** Let $x \in Z(N)$.

Then $Ann_N(x) \subseteq Ann_N(x^2) \subseteq \dots\dots\dots$ is an ascending chain of essential left ideals in N, since $Ann_N(x) \leq_e N$.

So for some $t \in I^+$, $Ann_N(x^{t+1}) \leq_e N$ by proposition 1.3.3.

We claim $x^t = 0$.

Suppose $x^t \neq 0$.

Then we get $Ann_N(x^{t+1}) \cap Nx^t \neq 0$.

As N has A.C.C. on essential left ideals $\exists\, t \in I^+$ such that $Ann_N(x^t) = Ann_N(x^{t+1})$, whence we get $Ann_N(x^{t+k}) = Ann_N(x^t)$ for all $k \in I^+$.

Let $y = n x^t (\neq 0) \in Ann_N(x^{t+1}) \cap Nx^t$ for $n \in N$.

Now $y \in Ann_N(x^t) \Rightarrow y x^t = 0$

$\Rightarrow n x^{2t} = 0$

$\Rightarrow n \in Ann_N(x^{2t}) = Ann_N(x^t)$

$\Rightarrow y = n x^t = 0$, a contradiction.

i.e. $y \in Ann_N(x^{t+1}) \Rightarrow y \notin Ann_N(x^t) \Longrightarrow Ann_N(x^t) \neq Ann_N(x^{t+1})$, a contradiction.

Thus Z(N) contains nilpotent elements.

As finite intersection of essential N-subgroups of N is distributively generated, Z(N) is N-subgroup of N. [ by proposition 2.1.14]

So $J_2(N)$ contains Z(N).

By lemma 3.4.20, Z(N)=0.

For $S^3I$ near-ring N, N/Soc(N) is weakly Noetherian by proposition 3.4.5. Again from proposition 3.4.9, (considering N as N-group) it follows that N has acc on essential ideals when we get N is non singular.

**Theorem 3.4.22:** If $\{N\bar{e}\}_{\bar{e}\in\frac{N}{Soc(N)}}$ is an independent family of normal N-subgroups of N/Soc(N)-group E, direct sum of E-injective N/Soc(N)-groups is commutative N-group, then $N/_I$ is weakly Noetherian $V_c$ N-group for every essential ideal I of N implies $N/_{Soc(N)}$ is weakly Noetherian $V_c$ near-ring.

**Proof:** $N/_I$ is weakly Noetherian for every essential ideal I of N implies $N/_{Soc(N)}$ is weakly Noetherian as proposition 3.4.8.

Let L be a strictly semi-simple N/Soc(N)-group.

Then as N dgnr, L is a semi-simple N/Soc(N)-group.

I/ Soc(N) an ideal of N/Soc(N) and f : I/ Soc(N) → L a non-zero N-homomorphism.

Let Kerf = K/ Soc(N).

Now K is essential in N. For if K ∩ J = 0 for some non-zero ideal J of N then $J \cong \frac{J+K}{K}$ and since the latter is isomorphic to a ideal of L, it follows that for some ideal $I_1 \neq 0$ and contained in J that $I_1 \subseteq L$, hence $I_1 \subseteq$ Soc(N) ⊆ K, a contradiction.

Thus N/K is a weakly Noetherian $V_c$ N-group.

If N → N/ Soc(N) is canonical quotient map, then (N/ Soc(N))/( K/ Soc(N)) is a weakly Noetherian $V_c$ N-group. Proposition 3.4.2, yields a map of $\frac{N}{Soc(N)}$ into L. So, L is $\frac{N}{Soc(N)}$-injective.

Thus by corollary 3.4.4, $^{N}/_{Soc(N)}$ is weakly Noetherian $V_c$ near-ring.

*If every injective right N/K-group is injective as an N-group we get the following result.*

**Theorem 3. 4.23:** For a near-ring N with unity the following conditions are equivalent:

i. N is $S^2S_wI$ -near-ring.

ii. $^{N}/_{Soc(N)}$ is weakly Noetherian $V_c$ near-ring.

**Proof:** i. ⇒ii. By corollary 3.4.4, we have to show that every strictly semi-simple N/Soc(N)-group E is injective .

If E is N/Soc(N)-group then SocN.E = 0.

Now Ann(E) = { x ∈ N / xE = 0} is essential in N.

Again as Soc(N).E = 0, Soc(N) ⊆ Ann(E). Thus Soc(N) = Ann(E) , that is E is annihilated by Soc(N). Again $Z_w(E) = \{x \in E / Ix = 0, I \leq_{ei} N\}$ and we get E is weak singular.

For if not for some $x \in E$, $\forall I \leq_{ei} N$ , $Ix \neq 0$, that is SocN.x $\neq 0$,a contradiction.

By( i.) E is injective as an N-group and hence injective as an N/Soc(N)-group.

ii.⇒i. Let L be a semi-simple weak singular N-group.

Then L can be regarded as N/ Soc(N)-group and hence injective as N/ Soc(N)-group by (ii).

So L is injective as N-group.

*For near-ring N with identity and M unital N-group if for every right ideal U of N and every N-homomorphism f : U ⟶ M, there exists an element m in M such that f(a) = ma for all a in U implies M is injective then we get the following results.*

**Proposition 3. 4.24:** $\oplus_{\iota \in I} E_i$ of injective N-groups is injective if near-ring N is weakly Noetherian.

**Proof:** Let N be weakly Noetherian, I be an ideal of N and f: $I \longrightarrow \oplus_A E_\alpha$ .

Then since I is finitely generated, *Imf* is contained in $\oplus_F E_\alpha$ for some finite subset $F \subseteq A$.

So $\oplus_F E_\alpha$ is injective since finite direct sum is injective by theorem 3.3.7.

By theorem 4.1.9, as $\oplus_F E_\alpha$ is injective then for every right ideal U of N and every N-homomorphism $f : U \longrightarrow \oplus_F E_\alpha$, there exists an element m in $\oplus_F E_\alpha$ such that f(a) = ma for all $a \in U$. But $m \in \oplus_A E_\alpha$ also. So for every right ideal U of N and every N-homomorphism $f : U \longrightarrow \oplus_A E_\alpha$, there exists an element m in $\oplus_A E_\alpha$ such that f(a) = ma for all a in U. Then $\oplus_A E_\alpha$ is injective.

**Proposition 3.4.25**: For any near-ring N the following conditions are equivalent:

i. N is an almost weakly Noetherian near-ring.

ii. N/I is weakly Noetherian for every essential left ideal I of N.

iii. N has A.C.C. on essential left ideals.

Moreover if Z ($_NN$) = 0 , N dgnr and every injective right N/K-group is injective as an N-group for ideal K of N we get

iv. Direct sum of (countably many) weak singular injective left N-groups is injective.

Again if Z ($_NN$) = 0 and every injective right N-group is injective as an N/K-group for ideal K of N where SocN is pure we get

v. Direct sum of (Countably many) injective hulls of simple weak singular left N-groups is injective.

**Proof:** Equivalence between (i), (ii), (iii) is clear from above corollary 3.4.13, considering N as N-group.

(i) ⇒ (iv). Let $\{E_i\}_{i\in I}$ be a family of weak singular left N-groups. Since $Z_w(E_i) = \{x \in E_i$ / $Ix = 0$ for $I \leq_{ei} N\} = E_i$ ,we get SocN.$E_i$ = 0. So each $E_i$ can be regarded as an N/Soc(N)-groups. Since N/Soc(N) is weakly Noetherian, $\oplus_{i\in I}E_i$ is injective as an N/Soc(N)-group by proposition 3.4.24, hence $\oplus_{i\in I} E_i$ is injective as an N-group.

(iv) ⇒(v). clear.

(v) ⇒(i). Proposition 3.4.17

*If every injective right N/K-group is injective as an N-group we get the following results:*

**Theorem 3.4.26:** For a dgnr near-ring N, then the following conditions are equivalent:

i. N is $S^2S_wI$ -near-ring.

ii. $^N/_{Soc(N)}$ is weakly Noetherian $V_c$ near-ring.

iii. N is GV-near-ring and direct sum of weak singular injective N-groups is injective.

iv. N is GV-near-ring and N has A.C.C. on essential left ideals.

**Proof:** i.⇔ii. From theorem 3.4.23

ii.⇒iii. From equivalence between (i) and (ii) clearly N is a GV-near-ring.

N/ Soc(N) is weakly Noetherian.

Let $\{E_i\}_{i\in I}$ be a family of weak singular left N- groups. Clearly each $E_i$ can be regarded as an N/Soc(N)-groups.

Since N/Soc(N) is weakly Noetherian, so by proposition 3.4.24, $\oplus_{i\in I}E_i$ is injective as an N/Soc(N)- group. So $\oplus_{i\in I}E_i$ is injective as an N-group.

iii.⇒i. is obvious.

**Theorem 3.4.27:** For a dgnr GV near-ring N direct sum of weak singular injective N-groups is injective implies N has A.C.C. on essential left ideals.

**Proof:** Since (iii) is equivalent to (ii) in theorem 3.4.26, we can conclude that N has A.C.C. on essential ideals.

**Theorem 3.4.28:** For a dgnr GV near-ring N if every injective right N/K-group is injective as an N-group for ideal K of N and N has A.C.C. on essential left ideals then direct sum of weak singular injective N-groups is injective.

**Proof:** From theorem 3.4.21, Z(N) = 0.

From proposition 3.4.25 direct sum of weak singular injective N-groups is injective.