Chapter 2

HONESTY, SUPERHONESTY IN NEAR-RINGS AND NEAR-RING GROUPS

2. HONESTY, SUPERHONESTY IN NEAR-RINGS AND NEAR-RING GROUPS

In this chapter we discuss the notions honesty and superhonesty in near-rings and near-ring groups. The chapter is divided into four sections. Some of the contents of this chapter form the papers [57], which is published in Indian Journal of Mathematics and Mathematical Science and [58], which is published in Advances in algebra.

2.1. PRELIMINARIES:

This section contains some basic definitions and results which are used in the sequal. Considering χ as the set of all essential N-subgroups ($0 \notin \chi$) of N, we define χ-honest, χ-closed, χ-torsion, torsion, superhonest N-subgroups and discuss some examples.

Definition 2.1.1: Let χ be the set of all essential N-subgroups such that $0 \notin \chi$ of near ring N. Let $K \subseteq E$ be an N-subgroup of an N- group E. We say K is χ -closed N- subgroup of E or K is χ-closed in E, if for any $I \in \chi$ and any $x \in E$, if $Ix \subseteq K$, then $x \in K$.

Example 2.1.2: Let $N = Z_6$ be a set with operations '+' addition module 6 defined as example 1.5.1 and '.' defined by following table:

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	4	0	4	1
2	0	2	2	0	2	2
3	0	0	0	0	0	3
4	0	4	4	0	4	4
5	0	2	2	0	2	5

Then (Z_6, +, .) is a near-ring. If A = {0, 3} then A is N-subgroup of $_NN$. Since N ∈ χ, Ne ⊆ A for e = 0 and 3. So A is χ-closed.

Let Z be the near-ring of integers. Then 3Z is an essential N-subgroup of Z.

Now 3Z.x ∈2Z implies x ∈ 2Z. So the N-subgroup 2Z of Z is χ-closed.

Definition 2.1.3: Let χ be the set of all essential N-subgroups such that $0 \notin \chi$ of near ring N. Let $K \subseteq E$ be an N-subgroup of an N- group E. We say K is χ-honest N-subgroup of E or K is χ -honest in E, if for any I ∈ χ and any $x \in$ E, if Ix $(\neq 0) \subseteq K$, then x ∈ K.

Example 2.1.4: Let N = Z_6 is a set with operations '+' as addition modulo 6 defined as example 1.5.1 and '.' defined by following table:

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

Then (Z_6, +, .) is a near-ring.

If A = {0, 3} then A is N-subgroup of $_NN$. For N ∈ χ , Ne $(\neq 0)$ ⊆ A for e = 0 and 3.

So A is χ-honest.

Note 2.1.5: If K is χ-closed in E, then K is χ-honest in E.

Definition 2.1.6: The set $(B : a) = \{ n \in N / na \in B\}$.

Proposition 2.1.7: If B is an essential N-subgroup of E and $a \in E$ then $(B : a) \in \chi$.

Proof: Let $x, y \in (B : a) \Rightarrow xa, ya \in B$

$\Rightarrow xa - ya \in B$

$\Rightarrow (x - y)a \in B$

$\Rightarrow (x - y) \in (B : a)$

Next $y \in (B : a) \Rightarrow ya \in B$

For any $n \in N$, $n(ya) \in NB \subseteq B$ [since B is N-subgroup of E].

So $(ny)a \in B \Rightarrow ny \in (B : a)$.

Thus (B : a) is N-subgroup of N.

Now $B \leq_e E$, Let K be nonzero N-subgroup of N.

Since $a \in E$, Ka is N-subgroup of E.

$Ka = (0) \Rightarrow ka = 0 \in B, \forall k (\neq 0) \in K$

$\Rightarrow k \in K \cap (B : a)$

$\Rightarrow K \cap (B : a) \neq 0$

$Ka \neq (0) \Rightarrow Ka \cap B \neq 0$

Let $k_1a (\neq 0) \in B$, for $k_1 \in K$.

Then $k_1 \in (B : a) \Rightarrow k_1 \in K \cap (B : a)$

If $k_1 = 0$, $k_1 a = 0$, a contradiction.

So $k_1 \neq 0$, which gives $K \cap (B : a) \neq 0$.

Thus $(B : a) \leq_e N \Rightarrow (B : a) \in \chi$.

Corollary 2.1.8: If $B \in \chi$ and $a \in N$ then $(B : a) \in \chi$.

Definition 2.1.9: χ-torsion of N-group E is the subset $\{ e \in E \,/\, Ie = 0$ for some N-subgroup I of $\chi\}$ and is denoted by $T_\chi(E)$.

Definition 2.1.10: If $T_\chi(E) = E$, E is called χ-torsion N-group.

Example 2.1.11: $N = Z_8$ is the set with two operations '+' as addition modulo 8 and '.' defined by following table. Then $(Z_8, +, .)$ is a near-ring.

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	0	4	0	4	0	4
2	0	0	0	0	0	0	0	0
3	0	4	0	4	0	4	0	4
4	0	0	0	0	0	0	0	0
5	0	4	0	4	0	4	0	4
6	0	0	0	0	0	0	0	0
7	0	4	0	4	0	4	0	4

Here I = {0, 2, 4, 6} $\in \chi$ and Ie = 0, for e = 0, 1, 2, 3, 4, 5, 6, 7. So $T_\chi(E)$ = E, E is χ-torsion N-group.

Definition 2.1.12: If $T_\chi(E)$ = 0, E is called χ-torsion free N-group.

Example 2.1.13: N = Z_8 is the set with two operations '+' as addition modulo 8 and '.' defined by following table. Then (Z_8, +, .) is a near-ring.

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1
2	0	2	2	2	2	2	2	2
3	0	3	3	3	3	3	3	3
4	0	4	4	4	4	4	4	4
5	0	5	5	5	5	5	5	5
6	0	6	6	6	6	6	6	6
7	0	7	7	7	7	7	7	7

Here I = N$\in \chi$ and Ie = 0, for e = 0. So $T_\chi(E)$ = 0, E is χ-torsion free N-group.

Proposition 2.1.14: If proper essential N-subgroups of N are distributively generated, then $T_\chi(E)$ N-subgroup of E.

Proof: Let $e_1, e_2 \in T_\chi(E)$ $\Rightarrow \exists\ I_1, I_2 \in \chi$ such that $I_1e_1 = 0$, $I_2e_2 = 0$.

Now $I_1 \cap I_2 \in \chi$ and $(I_1 \cap I_2)e_1 = 0$, $(I_1 \cap I_2)e_2 = 0$.

Since $(I_1 \cap I_2)$ is distributively generated, $(I_1 \cap I_2)(e_1 - e_2) = 0$.

So $e_1 - e_2 \in T_\chi(E)$.

Again let $e \in T_\chi(E)$, so $Ie = 0$ for $I \in \chi$.

To show for $a \in N$, $ae \in T_\chi(E)$.

Now for $I \in \chi$, $(I : a) \in \chi$, for $a \in N$.

Let $z \in (I : a)$ then $za \in I$.

We get $(za)e = 0$ [since $Ie = 0$]

$\Rightarrow z(ae) = 0$

So, $(I : a) \in \chi$ such that $z \in (I : a)$ and $ae \in T_\chi(E)$.

Thus $T_\chi(E)$ is N-subgroup of E.

Definition 2.1.15: χ-closure of M in E is the subset $\{e \in E \mid Ie \subseteq M$, for some N-subgroup I of $\chi\}$ and we denote it by $Cl_\chi^E(M)$ or simply $Cl_\chi(M)$.

So M is χ-closed if $Cl_\chi^E(M) = M$.

Note 2.1.16: (i) Let M be an N-subgroup of E. Then $M \subseteq Cl_\chi^E(M)$.

[Since $Cl_\chi^E(M) = \{ x \in E / \exists I \in \chi$ s.t. $Ix \subseteq M\}$ and M N-subgroup of E.

(ii) If $M_1 \subseteq M_2$ are N-subgroups of E, then $Cl_\chi^E(M_1) \subseteq Cl_\chi^E(M_2)$ for $M_1, M_2 \subseteq E$.

[Obvious from definition]

Example 2.1.17: In example 2.1.11, $M = \{0, 2, 4, 6\}$ N-subgroup of ${}_NN$. $I = \{0, 4\} \in \chi$. Now $Ie \subseteq M$ for all $e \in N$. So $Cl_\chi^N(M) = {}_NN$.

Definition 2.1.18: The set of torsion elements of E , $T_N(E) = \{ e \in E \ / \ (0,e) \neq 0\}$

Definition 2.1.19: If $T_N(E) = E$, E is called torsion N-group.

Example 2.1.20: N = {0, a, b, c} is the Klein's four group with multiplication

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

Then (N, +, .) is a near-ring. In $_N N$, $T_N(_N N) = {}_N N$. So $_N N$ is torsion N-group.

Definition 2.1.21: If $T_N(E) = 0$, E is called torsion-free N-group.

Example 2.1.22: N = {0, a, b, c} is the Klein's four group with multiplication

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

Then (N, +, .) is a near-ring. In $_N N$, $T_N(_N N) = 0$. So $_N N$ is torsion-free N-group.

Definition 2.1.23: An N-subgroup (ideal) M is called super-honest in E if $x \in E \setminus M$ for $n \in N$, $nx \in M \Rightarrow n = 0$.

If B is an N-subgroup (ideal) of N then B is called a super-honest N-subgroup (ideal) of N if B is super-honest N-subgroup (ideal) of N considered N as N-group $_N N$.

Example 2.1.24: Superhonest N-subgroups:

Here N = {0, 1, 2, 3, 4, 5}, N_2 = {0, 1}, N_3 = {0, 1, 2} are near-rings under the operation '+' as addition module 6, modulo 2, modulo 3 respectively and the multiplication '*' defined as

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	2	2	2	2	2
3	0	3	3	3	3	3
4	0	4	4	4	4	4
5	0	5	5	5	5	5

Now $N_2 \oplus N_3 \oplus N$ is a group.

We define the map $N \times (N_2 \oplus N_3 \oplus N) \to N_2 \oplus N_3 \oplus N$

by $n(a,b,c) = (n{*}a, n{*}b, n{*}c)$ for all $n \in N$, $(a, b, c) \in N_2 \oplus N_3 \oplus N$,

where $n{*}a \in N_2$ and $n{*}b \in N_3$.

Then $N_2 \oplus N_3 \oplus N$ is an N-group.

Now $N_2 \oplus N_3$ is an N-subgroup of $N_2 \oplus N_3 \oplus N$.

For $(a_1, b_1, c_1) \in (N_2 \oplus N_3 \oplus N) - (N_2 \oplus N_3)$ implies $c_1 \neq 0$.

If for some $n \in N$, $n(a_1, b_1, c_1) \in (N_2 \oplus N_3)$

$\Rightarrow (n^*a_1, n^*b_1, n^*c_1) \in (N_2 \oplus N_3)$

$\Rightarrow (n^*a_1, n^*b_1, n) \in (N_2 \oplus N_3)$ [since $c_1 \neq 0$]

$\Rightarrow n = 0$.

So $N_2 \oplus N_3$ is a superhonest N-subgroup of $N_2 \oplus N_3 \oplus N$.

Definitions 2.1.25: An N-subgroup M of E is called essentially closed if whenever C is an N-subgroup of E such that $M \subseteq_e C$ then $C = M$.

An N-subgroup M of E is called weakly essentially closed if whenever C is an N-subgroup of E such that such that $M \subseteq_{we} C$ then $C = M$.

2.2 CHARACTERSTICS OF χ - HONEST N-SUBGROUPS:

In this section we characterize χ-honest N-subgroups using the concepts like χ-closed, χ-torsion.

Lemma 2.2.1: Let $H \subseteq M \subseteq E$ be N-subgroups, then the following statements hold.

(a) If H is χ-honest in M and M is χ-honest in E then H is χ-honest in E.

(b)If M is χ-honest in E if and only if M/ H is χ-honest in E/H, where H is ideal of E.

(c) If H is ideal of E and H, M/ H are χ-honest in E/H, then M is χ-honest in E.

Proof: (a) Let e $\in$ E, I $\in \chi$ be such that Ie($\neq$ 0) $\subseteq$ H, then Ie $\subseteq$ M [since H $\subseteq$ M] hence e $\in$ M as M is χ-honest in E.

But e $\in$ M, Ie $\subseteq$ H $\Rightarrow$ e $\in$ H as H is χ-honest in M. So e $\in$ E , I $\in \chi$ such that Ie $\subseteq$ H $\Rightarrow$ e $\in$ H gives H is χ-honest in E.

(b) Let e $\in$ E , I $\in \chi$ be such that I(e + H)($\neq$ 0) $\subseteq$ $M/_H$, then Ie ($\neq$ 0) $\subseteq$ M. Hence e $\in$ M.

We get e + H $\in$ $M/_H$. So M/ H is χ-honest in E/H.

(c) Let e $\in$ E, I $\in \chi$ be such that Ie ($\neq$ 0)$\subseteq$ M. If Ie $\subseteq$ H then e $\in$ H $\subseteq$ M. If Ie $\not\subseteq$ H, then

(0 $\neq$)I(e + H) $\in$ $M/_H$, hence (e+ H) $\in$ $M/_H$ and we get e $\in$ M.

Similarly we get the same properties for χ-closed N-subgroups also.

Lemma 2.2.2: Let $\{M_\lambda : \lambda \in \Lambda\}$ be a family of χ-honest N-subgroups of E, then $\cap_\lambda M_\lambda$ is χ-honest.

Proof: Let e $\in$ E , I $\in$ χ be such that Ie($\neq$ 0) $\subseteq \cap_\lambda M_\lambda$, $\therefore$ Ie $\subseteq M_\lambda$, $\forall \lambda$

$\Rightarrow$ e $\in M_\lambda$ [$\because M_\lambda$ χ closed] $\forall \lambda$

$\Rightarrow$ e $\in \cap_\lambda M_\lambda$

Similarly we get the lemma for χ-closed N-subgroups also.

Lemma 2.2.3: If proper essential N-subgroups of N are distributively generated then for any left N-group E and any N-subgroup M $\subseteq$ E we have $Cl_\chi^E(M)$ is an N-subgroup of E.

Proof: Let $X_1, X_2 \in Cl_\chi^E(M)$, then there exist $I_1, I_2 \in \chi$ such that $I_iX_i \subseteq M$ for $i=1,2$.

Since $I_1 \cap I_2 \in \chi$ and $I_1 \cap I_2$ is distributively generated.

So, $(I_1 \cap I_2)(X_1 + X_2) \subseteq M$.

Then $X_1 + X_2 \in Cl_\chi^E(M)$

Let $x \in Cl_\chi^E(M)$ and $n \in N$, then there exist $I \in \chi$ such that $Ix \subseteq M$.

since $I \in \chi$, we have $J = (I : n) \in \chi$ i.e. $Jn \subseteq I$.

So we have $Jnx \subseteq Ix \subseteq M$, hence $nx \in Cl_\chi^E(M)$.

Thus $Cl_\chi^E(M)$ is an N-subgroup of E.

Lemma 2.2.4: $Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2) = Cl_\chi^E(M_1 \cap M_2)$ for any N-subgroups $M_1, M_2 \subseteq E$ and N-group E.

Proof: We always have $Cl_\chi^E(M_1 \cap M_2) \subseteq Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2)$

$\because x \in Cl_\chi^E(M_1 \cap M_2) \Longrightarrow \exists I \in \chi$ such that $Ix \subseteq M_1 \cap M_2$ i.e. $Ix \subseteq M_1, Ix \subseteq M_2$

i.e. $x \in Cl_\chi^E(M_1)$, $x \in Cl_\chi^E(M_2) \Longrightarrow x \in Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2)$.

Otherwise if $x \in Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2)$, there exists $I_1, I_2 \in \chi$, such that $I_ix \subseteq M_i$ then $I_1 \cap I_2 \in \chi$ hence $I_1 \cap I_2, x \subseteq M_1 \cap M_2$ and $x \in Cl_\chi^E(M_1 \cap M_2)$.

Definition 2.2.5: The set χ of essential N-subgroup is called linear filter if $I \subseteq N$ and $J \in \chi$ satisfy $(I : y) \in \chi$ for any $y \in J$, then $I \in \chi$. It is denoted by $\mathcal{L}$.

Proposition 2.2.6: Let $\mathcal{L}$ be the set of essential N-subgroups, then the following statements are equivalent:

(a) $\mathcal{L}$ is a linear filter .

(b) $Cl_{\mathcal{L}}^{E}\mathrm{Cl}_{\mathcal{L}}^{E} = \mathrm{Cl}_{\mathcal{L}}^{E}$ for any N-group E.

Proof: (a) ⟹(b): If M ⊆ E is an N-subgroup and x ∈ $Cl_{\mathcal{L}}^{E}\mathrm{Cl}_{\mathcal{L}}^{E}$(M) , ∃ I ∈ $\mathcal{L}$ such that Ix ⊆ $\mathrm{Cl}_{\mathcal{L}}^{E}$(M) , then to any y ∈ I ∃ I_y ∈ $\mathcal{L}$ such that $I_y$yx ⊆ M , $I_y y$ ⊆ (M : x) i.e. for any $I_y$yx ⊆ M, hence ((M : x) : Y) belongs to $\mathcal{L}$, therefore (M : x) ∈ $\mathcal{L}$ as $\mathcal{L}$ is a linear filter.

⟹ x ∈ $\mathrm{Cl}_{\mathcal{L}}^{E}$M) ⟹ $Cl_{\mathcal{L}}^{E}\mathrm{Cl}_{\mathcal{L}}^{E}$M) ⊆ $\mathrm{Cl}_{\mathcal{L}}^{E}$(M)

$\mathrm{Cl}_{\mathcal{L}}^{E}$(M) ⊆ $Cl_{\mathcal{L}}^{E}\mathrm{Cl}_{\mathcal{L}}^{E}$(M) is obvious by definition of χ-closure.

Thus $\mathrm{Cl}_{\mathcal{L}}^{E}$M) = $Cl_{\mathcal{L}}^{E}\mathrm{Cl}_{\mathcal{L}}^{E}$(M).

(b) ⟹ (a): Let J ∈ $\mathcal{L}$ and I ⊆ N be such that for any y ∈ J the N-subgroup (I : y) ∈ $\mathcal{L}$,

Hence N = $\mathrm{Cl}_{\mathcal{L}}^{E}$(J) ⊆ $Cl_{\mathcal{L}}^{E}\mathrm{Cl}_{\mathcal{L}}^{E}$(I) = $\mathrm{Cl}_{\mathcal{L}}^{E}$(I)

∴ $\mathrm{Cl}_{\mathcal{L}}^{E}$(I) = N and we have I ∈ $\mathcal{L}$.

Lemma 2.2.7: Let M ⊆ E be an N-subgroup then the following statements are equivalent:

a. M is χ-honest in E.
b. For m ∈ (Cl_{χ}^{E}(M) \ M we have (M : m)= Ann (m)
c. For m ∈ (Cl_{χ}^{E}(M) \ M we have Nm ∩ M =0

Proof: (b) ⇒ (c) Let for some x ∈ (Cl_{χ}^{E}(M) \ M, (M : x) = Ann (x)

To show Nx ∩ M = 0

Let P (≠ 0) ∈ Nx ∩ M ⇒ P ∈ Nx and P ∈ M

⇒ P = nx for some n ∈ N and P ∈ M

i.e. $nx \in M$. i.e. $n \in (M : x)$

But $(M : x) = \mathrm{Ann}(x)$.

$\therefore n \in \mathrm{Ann}(x) \Longrightarrow nx = 0 \Rightarrow P = 0$, a contradiction.

(c) $\Rightarrow$(a) For any $x \in (Cl_{\chi}^{E}(M) \setminus M$ and $Nx \cap M = 0$, to show M is χ-honest in E.

Let for some $I \in \chi$ and $e \in E$. $0 \neq Ie \subseteq M$ to show $e \in M$.

If possible $e \notin M$. But $e \in Cl_{\chi}^{E}(M)$

$\therefore Ne \cap M = 0 \Rightarrow Ie \cap M = 0$. Which is a contradiction, since $Ie \subseteq M$.

$\therefore e \in M \Rightarrow M$ is χ- honest in E.

(a) $\Rightarrow$ (b) Let $x \in (Cl_{\chi}^{E}(M) \setminus M$.

Then there exists $I \in \chi$ with $Ix \subseteq M$, then $Ix = 0$ [$\because$ M is honest in E, $x \notin M$]

Hence $I \subseteq \mathrm{Ann}(x) \subseteq (M : x)$, therefore $(M : x) \in \chi$.

Hence $(M : x) = \mathrm{Ann}(x)$.

Lemma 2.2.8: Let $M \subseteq E$ be an χ-honest N- subgroup, then $Cl_{\chi}^{E}(M) = M \cup T_{\chi}(E)$

Proof: Let $x \in Cl_{\chi}^{E}(M)$

$\because M \subseteq Cl_{\chi}^{E}(M)$, $\therefore$ if $x \in M$ done.

If $x \notin M$ as $x \in Cl_{\chi}^{E}(M)$, $\exists\, I \in \chi$ such that $0 = Ix \subseteq M$ [$\because$ M is χ-honest in E]

$\Longrightarrow x \in T_{\chi}(E) \Longrightarrow x \in M \cup T_{\chi}(E)$

Conversely let $x \in M \cup T_{\chi}(E)$

$\Longrightarrow$ if $x \in M$ then $x \in Cl_{\chi}^{E}(M)$ obvious.

$x \in T_{\chi}(E) \Rightarrow \exists\, I \in \chi$ such that $Ix = 0 \in M \Rightarrow x \in Cl_{\chi}^{E}(M)$

$\therefore Cl_{\chi}^{E}(M) = M \cup T_{\chi}(E)$

Corollary 2.2.9: Let $M \subseteq E$ be an N-subgroup such that $T_{\chi}(E) \subseteq M$, then $M \subseteq E$ is χ-honest N-subgroup if and only if it is χ-closed.

In particular, if E is χ- torsion free then an N subgroup $M \subseteq E$ is χ-honest if and only if it is χ- closed.

Proof: By lemma 2.2.8 if $M \subseteq E$ is χ-honest then $Cl_{\chi}^{E}(M) = M \cup T_{\chi}(E) = M$ [as $T_{\chi}(E) \subseteq M$]

$\therefore$ M is χ -closed.

Conversely, M is χ -closed $\Rightarrow \chi$ honest obvious.

In particular E is χ – torsion free $\Rightarrow T_{\chi}(E) = 0$

So $M \subseteq E$ is χ- honest $\Rightarrow Cl_{\chi}^{E}(M) = M \cup T_{\chi}(E) = M$.

Thus M is χ-closed in E.

Corollary 2.2.10: Let $\mathcal{L}$ be a linear filter, then for any N-group E the torsion N-subgroup $T_{\mathcal{L}}(E)$ is $\mathcal{L}$-honest.

Proof: Since $\mathcal{L}$ is a linear filter then $T_{\mathcal{L}}(E)$ is a $\mathcal{L}$-closed N-subgroup, hence $\mathcal{L}$-honest.

[$\because T_{\mathcal{L}}(E) = Cl_{\mathcal{L}}^{E}(0)$].

$\therefore \mathcal{L}$-linear gives $Cl_{\mathcal{L}}^{E}Cl_{\mathcal{L}}^{E}(0) = Cl_{\mathcal{L}}^{E}(0)$ which gives $Cl_{\mathcal{L}}^{E}(0)$ is $\mathcal{L}$-closed.

i.e. $T_{\mathcal{L}}(E)$ is $\mathcal{L}$-honest.

Remark 2.2.11: $\mathcal{L}$ is a linear filter if and only if $T_{\mathcal{L}}(E) \subseteq E$ is $\mathcal{L}$-closed for any N-group E if and only if $T_{\mathcal{L}}(E) \subseteq E$ is $\mathcal{L}$-honest for any N-group E.

Corollary 2.2.12: Any χ-honest N-subgroup $M \subseteq E$ satisfies either $M \subseteq T_{\chi}(E)$ or $T_{\chi}(E) \subseteq M$, if proper essential N-subgroups of N are distributively generated.

Proof: Since M is χ-honest in E. So $Cl_{\chi}^{E}(M) = M \cup T_{\chi}(E)$.

$Cl_{\chi}^{E}(M)$ & $T_{\chi}(E)$ are subgroups.

Hence either M is included in $T_{\chi}(E)$ or $T_{\chi}(E)$ is included in M, as union of two subgroups is subgroup if one contain the other.

Note 2.2.13: If $\mathcal{L}$ is a linear filter and M is $\mathcal{L}$-honest in $T_{\mathcal{L}}(E)$, then M is $\mathcal{L}$-honest in E.

$\because$ $\mathcal{L}$ linear filter $\Longrightarrow$ $T_{\mathcal{L}}(E)$ is $\mathcal{L}$-honest.

$\therefore$ M is $\mathcal{L}$-honest in $T_{\mathcal{L}}(E)$ and $T_{\mathcal{L}}(E)$ is $\mathcal{L}$-honest in E $\Longrightarrow$ M is $\mathcal{L}$-honest in E.

Corollary 2.2.14: $(\neq 0)$ $M \subseteq E$ is χ-honest and if M is χ-torsion free then E is χ-torsion free and $(\neq 0)$ $M \subseteq E$ is χ-closed if proper essential N-subgroup of N is distributively generated.

Since $(\neq 0)$ $M \subseteq E$ is χ-honest so by corollary 2.2.12 either $M \subseteq T_{\chi}(E)$ or $T_{\chi}(E) \subseteq M$.

First to show $T_{\chi}(E) = 0$, if M is χ-torsion free.

Let $x (\neq 0) \in T_{\chi}(E) \Longrightarrow \exists\, I \in \chi$ such that $Ix = 0$.

Now if $x \in M$, $Ix \neq 0$, $\forall\, I \in \chi$ [$\because$ M is χ-torsion free], a contradiction.

Again let $(\neq 0)$ $x \notin M$, so we get $M \subseteq T_{\chi}(E)$

For this condition for any y $\in$ M $\Longrightarrow$ y $\in T_\chi$ (E)

$\Longrightarrow$ Jy = 0 for some J $\in \chi$.

But M is χ- torsionfree, a contradiction.

$\therefore$ x = 0 $\Longrightarrow$ E is χ -torsion free.

Next to show M $\subseteq$ E is χ –closed.

Let x $\in$ E, I $\in \chi$ such that Ix $\subseteq$ M.

Now E is χ-torsion-free, Ix $\neq$ 0.

Again M is χ-honest in E, so Ix ($\neq$ 0) $\subseteq$ M $\Longrightarrow$ x $\in$ M

$\Longrightarrow$ M $\subseteq$ E is χ-closed.

2.3 CHARACTERSTICS OF SUPERHONEST N-SUBGROUPS:

This section contains some properties of superhonest N-subgroups. The concepts like essentially closed, torsion are used to discuss various characteristics of superhonest N-subgroups. We also attempt to find some relation of χ-honest and superhonest N-subgroups.

Lemma 2. 3.1: Let M be an N-subgroup (ideal) of E. Then M is a super-honest N-subgroup (ideal) of E if and only if for each a $\in$ E, (M : a) is a super-honest N-subgroup (ideal) of N.

Proof: Let M be a super-honest N-subgroup of E.

If n $\in$ N is such that n $\notin$(M : a) with $n'n \in$ (M : a) for some $n' \in$ N then n' n a $\in$ M.

Since M is a super-honest N-subgroup of E, we have $n' = 0$.

Hence (M : a) is a super-honest N-subgroup of N.

Conversely let (M: a) be a super-honest N-subgroup of N.

If $a \in E$ is such that $a \notin M$ with $na \in M$ for some $n \in N$ then $1 \notin (M : a)$.

This implies $n.1 = n \in (M : a)$.

Since (M : a) is a super-honest N-subgroup of N, so n= 0.

Hence M is a super-honest N-subgroup of E.

Lemma 2. 3.2: Let M be an N-subgroup (ideal) of E. Then M is a super-honest N-subgroup (ideal) of E if and only if $(M: a) = 0$ for each $a \in E - M$.

Proof: Let M be a super-honest N-subgroup (ideal) in E. Then for each $x \in E - M$, $n \in N$, $nx \in M$ implies $n = 0$, this gives $(M: x) = 0$, for each $x \in E - M$.

On the other hand let $(M: x) = 0$ for each $x \in E - M$.

If for some $n \in N$, $nx \in M$ then $n \in (M : x)$.

This implies n = 0. This gives M is super-honest in E.

Lemma 2.3.3: $\{0\}$ is a super-honest N-subgroup of N if and only if N has no left zero divisors.

Proof: If N has no left zero divisors then $\{0\}$ is super-honest N-subgroup (ideal) of N.

[Since $n \in N$, $x \in N - 0$, $nx = o$ implies $n = 0$]

Let $\{0\}$ be a super-honest N-subgroup of N. If $n\ (\neq 0) \in N$ satisfying $n'n = 0$ for some n' $\in N$ then $n' = 0$. Thus N has no left zero divisors.

Example 2.3.4: Z_6 is a near-ring under the operation '+' as addition module 6 and the multiplication '*' defined as the following table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	2	2	2	2	2
3	0	3	3	3	3	3
4	0	4	4	4	4	4
5	0	5	5	5	5	5

Here {0} is superhonest N-subgroup of Z_6. Z_6 has no zero divisors.

Lemma 2.3.5: N-group E has a proper super-honest N-subgroup (ideal) M, then N has to has no left zero divisors.

Proof: M is super-honest N-subgroup (ideal) in E

$\Leftrightarrow$ (M : a) = 0 for $a \in E - M$ [by lemma 2.3.2]

$\Leftrightarrow$ 0 = (M : a) is super-honest N-subgroup (ideal) of N for $a \in E - M$ [by lemma 2.3.1]

$\Leftrightarrow$ N has no left zero divisors, where M is proper super-honest N-subgroup (ideal) of E.

Lemma 2.3.6: Let M be an N-subgroup of E. If M is a complement N-subgroup of some N-subgroup of E then M an essentially closed N-subgroup of E.

Proof: Suppose M is a complement N-subgroup of an N-subgroup C of E.

If there exists an N-subgroup D of E such that $M \subset D$ and M is an essential N-subgroup of D, then $D \cap C$ is a non zero N- subgroup of D.

But $(D \cap C) \cap M \subset C \cap M = 0$, a contradiction to the fact that M is an essential N-subgroup of D.

So M is an essentially closed N-subgroup of E.

Lemma 2.3.7: If M is an weakly essentially closed N-subgroup of E then M is a complement N-subgroup of M^c in E.

Proof: Suppose there exists an N-subgroup D of E such that $D \supset M$ and $D \cap M^c = 0$.

By given condition M is not weakly essential N-subgroup of D and so there exist a non zero ideal D′ of D such that $D' \cap M = 0$.

Then $D \cap (M^c + D') = D' + (M^c \cap D) = D'$.

Now $M \cap D' = M \cap D \cap (M^c + D')$

$\Rightarrow 0 = M \cap (M^c + D')$.

Therefore $M \cap (M^c + D') = 0$, which contradicts to the fact that M^c is a complement N-subgroup of M in E. So M is a complement N-subgroup of M^c in E.

Although every weakly essential N-subgroup is not essential, for some near-rings and N-groups every weakly essential N-subgroup is essential.

Example 2.3.8: If P is a division near ring, Z is the ring of integers then P X Z is a near ring with respect to componentwise addition & multiplication then every weakly essential P X Z subgroup of P X Z is essential.

N= {0, 1, 2, 3, 4, 5, 6, 7} is a near-ring under addition modulo 8 and multiplication defined as follows:

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Clearly, every weakly essential N-subgroup of N is essential.

If so we get the following:

From lemma 2.3.6 and lemma 2.3.7 we get the following:

Lemma 2.3.9 : If M is an N-subgroup of E and M^c is a complement N-subgroup of B in E, then the following statements are equivalent.

(i) M is essentially closed N-subgroup of E.

(ii) M is a complement N- subgroup of M^c in E

(iii) M is a complement N- subgroup of some N-subgroup of E.

Lemma 2.3.10: If M is an N-subgroup of E such that $T_\chi(E) \subseteq M$, then M is an essential N-subgroup of $Cl_\chi(M)$.

Proof: Let A be N-subgroup of $Cl_\chi(M)$.We assume $A \cap M = 0$.

Let $m \in A$, then $m \in Cl_\chi(M)$ implies $Im \subseteq M$, for some essential N-subgroup I of χ.

Also $Im \subseteq A$ implies $Im \subseteq M \cap A = 0$.

This gives $Im = 0$.

Thus $m \in T_\chi(E) \subseteq M$. So, $m \in A \cap M$.

$\therefore m = 0$. This gives A=0.

Thus M is an essential N-subgroup of $Cl_\chi(M)$.

If M is a essentially closed N-subgroup (ideal) of E such that $T_\chi(E) \subseteq M$, then by above we get $M = Cl_\chi(M)$. On the other hand if M is an χ-closed N-subgroup (ideal) of E then M is essentially closed N-subgroup of E and $T_\chi(E) \subseteq M$. For if M is an essential N-subgroup of C where C is N-subgroup of E then for each $x \in C$, $(M : x)$ is an essential N-subgroup of N, so belongs to χ, then $x \in Cl_\chi(M) = M$. Hence $C = M$. Thus M is a essentially closed N-subgroup of E. Hence we get the following lemma:

Lemma 2.3.11 : Let M be an N-subgroup of E. Then M is essentially closed N-subgroup of E satisfying $T_\chi(E) \subseteq M$ if and only if $Cl_\chi(M) = M$.

Corollary 2.3.12: If every weakly essential N-subgroup is essential in E then (1) M is essentially closed & $T_\chi(E) \subseteq M$, (2) M is a complement N-subgroup of M^c & $T_\chi(E) \subseteq M$,

(3) M is a complement N-subgroup of some N-subgroup of E & $T_\chi(E) \subseteq M$ (4) M is χ-closed are equivalent.

Lemma 2.3.13 : If M is an ideal of an N-group E then M is super-honest in E if and only if M is essentially closed in E , $T_\chi(E) \subseteq M$ and $T_\chi(E/M) \supseteq T_N(E/M)$.

Proof: Let C be an N-subgroup of E such that $M \subseteq_e C$. Then there exists $a \in C - M$ such that Na is a non zero N-subgroup of C. Since $Na \cap M \neq 0$, so $(M : a) \neq 0$, this contradicts that M is a super-honest N-subgroup of E. Thus M = C. Thus M is essentially closed. Again $a \in T_N(E) \Rightarrow (0 : a) \neq 0$

$\Rightarrow x (\neq 0) \in (0 : a)$ so $xa = 0$.

If $a \in M$ then it is done.

If $a \notin M$ i.e. $a \in E \setminus M$ then $x = 0$ [$\because$ M is super-honest in E].

Hence contradiction to $x \neq 0$. So $a \in M$.

Thus $T_N(E) \subseteq M$. And so $T_\chi(E) \subseteq M$, because $T_\chi(E) \subseteq T_N(E)$.

Now $T_N(E/M) = \bar{0}$. Since $T_N(E) = \{a \in E / (0 : a) \neq 0\}$.

So $T_N(E/M) = \{\bar{a} \in E/M : (\bar{0} : \bar{a}) \neq 0\}$.

Let $\bar{a} \in T_N(E/M)$, $a \notin M \Rightarrow (\bar{0} : \bar{a}) \neq 0$

$\Rightarrow \exists\, x (\neq 0)$ such that $x \in (\bar{0} : \bar{a})$

$\Rightarrow x\bar{a} = \bar{0}$

$\Rightarrow xa + M = M \Rightarrow xa \in M$ where $a \notin M$

$\Rightarrow x = 0$, $\because$ M is super-honest in E.

Therefore $\forall$ a $\in$ E\M, $(\bar{0}: \bar{a}) = 0$ and so $T_N(E/M) = \bar{0} = M$.

$\therefore$ $T_\chi(E/M) \supseteq T_N(E/M)$ holds trivially.

Conversely let a $\in$ E\ M with na $\in$ M for some n$\in$ N.

If $n \neq 0$, then $\bar{a} = a + M \in T_N(E/M)$.

[$\because$ $n\bar{a} = na + M \in M = \bar{0} \Rightarrow n \in (\bar{0}: \bar{a}) \Rightarrow \bar{a} \in T_N(E/M)$].

So $\bar{a} \in T_\chi(E/M)$. Thus $(\bar{0} : \bar{a}) = (M : a)$ belongs to χ.

So a $\in Cl_\chi(M) = M$, a contradiction [by lemma 2.3.11].

Proposition2.3.14: Let M be an ideal of an N-group E. If M is χ-closed in E and $T_\chi(E/M) \supseteq T_N(E/M)$ then M is complement N-subgroup of some torsion-free N-subgroup of E.

Proof: M is χ-closed N-subgroup of E $\Rightarrow$ M essentially closed N-subgroup of E.

So by Lemma 2.3.7 M is complement ideal of M^c in E[since essentially closed implies weakly essentially closed is obvious], where M^c is a complement of M in E.

It remains to show M^c is a torsion-free N-subgroup of E.

Suppose there exists $0 \neq a \in M^c$ such that na = 0 for some $0 \neq n \in N$.

Then $\bar{a} = a + M \in T_N(E/M)$ and so $\bar{a} \in T_\chi(E/M)$, which implies that $(\bar{0} : \bar{a}) = (M : a)$ belongs to χ.

Thus a $\in Cl_\chi(M) = M$. But then a $\in M \cap M^c = 0$, contradiction to $0 \neq a$.

Therefore M^c is torsion-free N -subgroup of E.

Corollary2.3.15: If M is an ideal of an N-group E and every weakly essential N-subgroup of E is essential then (1) M is super-honest in E (2) M is complement N-subgroup of some torsion-free N-subgroup of E and $T_\chi(E) \subseteq M$ and $T_\chi(E/M) \supseteq T_N(E/M)$ (3) M is χ-closed and $T_\chi(E/M) \supseteq T_N(E/M)$ (4) M is an essentially closed in E , $T_\chi(E) \subseteq M$ and $T_\chi(E/M) \supseteq T_N(E/M)$ are equivalent.

Corollary 2.3.16: Since $T_\chi(E/M) \subseteq T_N(E/M)$ is obvious, we have $T_\chi(E/M) \supseteq T_N(E/M)$ if and only if $T_\chi(E/M) = T_N(E/M)$. Again $T_\chi(E/M) \supseteq T_N(E/M)$ if and only if (M : a) ≠ 0 for some a ∈ E, then (M : a) is an essential N-subgroup of N. If $_NN$ is uniform i.e. intersection of two non-zero N-subgroups is non-zero then $T_\chi(E) = T_N(E)$ [$\because x \in T_N(E) \Rightarrow (0 : x) \neq 0$ $\Rightarrow$ intersection of (0 : x) with nonzero N-subgroup is non-zero $\Rightarrow$ (0 : x) is essential in N $\Rightarrow$ $x \in T_\chi(E)$].

Corollary 2.3.17: If $_NN$ is uniform and if M is an ideal of an N-group E and every weakly essential N-subgroup of E is essential then (1) M is super-honest in E (2) M is complement N-subgroup of some torsion-free N-subgroup of E and $T_\chi(E) \subseteq M$ (3) M is χ-closed (4) M is an essentially closed in E , $T_\chi(E) \subseteq M$ are equivalent.

Note2.3.18: (1) It is clear that N-group E is a super-honest N-subgroup of E itself.

(2) Again Every super-honest N-subgroup contains $T_N(E)$.

[Let M is a super-honest N-subgroup of E. $x \in T_N(E) \Rightarrow nx = 0, n \in N$.

If $x \in M$ it is done.

If $x \notin M$ i.e. $x \in E \setminus M$ then n = 0 as M is super-honest in E, a contradiction.

So $x \in M \Rightarrow T_N(E) \subseteq M$]

(3) It is also clear that if E is torsion N-group, then E is the only super-honest N-subgroup of E. [Since E is torsion N-group $\Rightarrow T_N(E) = E$. If M is super-honest ideal of E, then M contains $T_N(E)$. $\Rightarrow$ M contains E. So M = E]

Proposition 2.3.19: If for each $i \in I$, M_i is a super-honest N-subgroup (ideal) of N-group E, then $\cap M_i$, $i \in I$ is also a super-honest N-subgroup (ideal) of E.

Proof: Let $x \in E$, $x \notin \cap_{i\in I} M_i$, with $nx \in \cap_{i\in I} M_i$ for some $n \in N$.

Now let $x \in E$ and $x \notin M_i$ for some i at least and $nx \in M_i$ $\forall i \in I$

$\Rightarrow n = 0$ as M_i is super-honest in E

$\Rightarrow \cap_{i\in I} M_i$ is super-honest N-subgroup of E.

The intersection of all super-honest N-subgroups (ideals) of E is the smallest super-honest N-subgroup (ideal) of E. We denote it by P. If $P \subsetneq E$, then E has proper super-honest N-subgroups, otherwise E is the only super-honest N-subgroup of E itself.

Lemma 2.3.20 : If E and E′ are N groups, f is a N-homomorphism from E to E′, then for each super-honest N-subgroup B′ of E′, $f^{-1}(B')$ is a super-honest N-subgroup of E.

Proof: Let $a\in E - f^{-1}(B')$ with $na \in f^{-1}(B')$ for some $n\in N$. Then $f(a)\in E' - B'$ and $nf(a) = f(na) \in B'$. Since B′ is super-honest in E′, it follows that n = 0. Hence $f^{-1}(B')$ super-honest in E.

Corollary 2.3.21: If P is the smallest super-honest N-subgroup of an N-group E, then for each N-endomorphism f of E, $f^{-1}(P) \supset P \supset f(P)$.

Proof: Since $f^{-1}(P)$ is a super-honest N-subgroup of E, $f^{-1}(P) \supset P$. Hence $P \supset f(P)$.

As the smallest super-honest N-subgroup P of an N-group E, we know $P \supset Cl_\chi(D)$, where D is the N-subgroup of E generated by $T_N(E)$.

Note 2.3.22: But if B is super-honest in E, then the following example shows that f(B) is not super-honest in E.

As example 2.1.24 if E is the N-group $Z_2 \oplus Z_3 \oplus Z_6$ of near-ring Z_6 then $P = Z_2 \oplus Z_3$ is superhonest in $Z_2 \oplus Z_3 \oplus Z_6$. So $\pi(P) = Z_2$ is not super-honest in E, where π is the projection from E onto Z_2.

Proposition2.3.23: $T_\chi T_\chi(E) = Cl_\chi T_\chi(E)$ is χ-closed N-subgroup of E.

Proof: For any two N-subgroups define a relation ~ s.t. $M_1 \sim M_2 \Leftrightarrow M_1 \cap X = 0$ if and only if $M_2 \cap X = 0$, for any N-subgroup X of E, M_1, M_2 N-subgroups of E. Then for an N-subgroup M of E if $M \sim E$ then M is essential in E. In case E = N, M is essential N-subgroup of N.

Now we prove (1) $Cl_\chi(M) \sim M + Cl_\chi(0)$

Let X be an N-subgroup of E such that $(M + Cl_\chi(0)) \cap X = 0$

Let $m \in Cl_\chi(M) \cap X$.

Now $m \in Cl_\chi(M) \Rightarrow \exists\, A \in \chi$ such that $Am \subseteq M$ and $m \in X \Rightarrow Am \subseteq X$

$\therefore Am \subseteq M \cap X = 0$ [$\because 0 = (M + Cl_\chi(0)) \cap X \supseteq M \cap X$]

$\Rightarrow m \in Cl_\chi(0)$ and $m \in X \Rightarrow m \in Cl_\chi(0) \cap X = 0 \Rightarrow m = 0 \therefore Cl_\chi(M) \cap X = 0$.

(2) $P \sim M \Rightarrow P \subseteq Cl_\chi(M)$

Let $p \in P$. Consider $A = \{ x \in N / xp \in M \} = (M : P)$ essential N-subgroup of N.

$\therefore$ p $\in Cl_\chi$ (M).

Now $Cl_\chi(M) \sim M + Cl_\chi(0)$.

In place of M considering $Cl_\chi(M)$ we get $Cl_\chi Cl_\chi(M) \sim Cl_\chi(M) + Cl_\chi(0) = Cl_\chi(M)$

i.e. $Cl_\chi Cl_\chi(M) \sim Cl_\chi(M)$.

Again $Cl_\chi Cl_\chi Cl_\chi(M) \sim Cl_\chi Cl_\chi(M)$.

i.e. $Cl_\chi Cl_\chi Cl_\chi(M) \sim Cl_\chi Cl_\chi(M) \sim Cl_\chi(M)$ i.e. $Cl_\chi Cl_\chi Cl_\chi(M) \sim Cl_\chi(M)$.

So by (2), $Cl_\chi Cl_\chi Cl_\chi(M) \subseteq Cl_\chi Cl_\chi(M)$.

$Cl_\chi Cl_\chi Cl_\chi(M) \supseteq Cl_\chi Cl_\chi(M)$ is obvious.

$\therefore Cl_\chi Cl_\chi Cl_\chi(M) = Cl_\chi Cl_\chi(M)$.i.e. $Cl_\chi Cl_\chi(M)$ is χ-closed.

In particular $Cl_\chi Cl_\chi(0) = T_\chi\ T_\chi(E) = Cl_\chi\ T_\chi(E)$ is χ-closed.

Proposition 2.3.24: If ${}_N N$ is uniform, for each N-group E, $T_N(E) = T_\chi(E)$ and then every χ-closed N-subgroup of E is super-honest in E. In particular $T_\chi\ T_\chi(E)$ is a super-honest N-subgroup of E.

Proof: If ${}_N N$ is uniform, for each N-group E, $T_N(E) = T_\chi(E)$ by corollary 2.3.16. Then every χ-closed N-subgroup of E is super-honest in E by corollary 2.3.17. In particular $T_\chi\ T_\chi(E)$ is χ-closed N-subgroup of E, hence super-honest in E.

Proposition 2.3.25: If the N-group E has no proper super-honest N-subgroup, $P'$is the smallest super-honest N-subgroup of the N-group E', then for each N-homomorphism f from E into E', $f(E) \subset P'$.

Proof: By proposition2.3.20. $f^{-1}(P')$ is a super-honest N-subgroup of E. But E has no proper super-honest N-subgroup and so $f^{-1}(P') = E$. Then $f(E) \subset P'$.

Corollary 2.3.26 : If the N-group E has no proper super-honest N-subgroup, E' is a torsionfree N-group then only N-homomorphism from E into E' is the zero homomorphism.

Proof: Since E' is torsionfree, 0 is the smallest super-honest N-subgroup of E'.

Note 2.3.27: If M is an ideal of an N-group E then M is super-honest N-subgroup of E implies M is χ-honest [by corollary 2.2.9 and lemma 2.3.13].

2.4 Some special types of χ-honest and superhonest N-groups:

In this section, considering χ as a set of essential N-subgroups, we study various characteristics of χ-honest and superhonest N-subgroups.

Throughout this section by χ *we mean a non empty set of essential N-subgroups of near ring N.*

It is to be noted that that Lemma2.2.1 and Lemma 2.2.2 .hold for this set χ.

Moreover if χ is closed under intersection we get proposition 2.1.14.

Definition 2.4.1: χ is called weak closed under intersection if for any $I_1, I_2 \in \chi$ there exists $J \in \chi$ such that $J \subseteq I_1 \cap I_2$.

Definitions 2.4.2: χ is left N-closed if for any $n \in N$ and any $I \in \chi$, there is $J \in \chi$ such that $J n \subseteq I$. Thus for any element $n \in N$ and any $I \in \chi$ we have $(I: n) \in \chi$

χ is left E-closed if for any $a \in E$ and any N-subgroup B of E there is a $J \in \chi$ such that $J a \subseteq B$. Thus for any element $a \in E$ & any N-subgroup B of E we have $(B : a) \in \chi$

Lemma 2.4.3: If χ is weak closed under intersection and proper essential N-subgroups of N are distributively generated then for any N-group E and any N-subgroup M of E , $Cl_\chi^E(M)$ is a subgroup of E.

Proof: Let $X_1, X_2 \in Cl_\chi^E(M)$, then there exist $I_1, I_2 \in \chi$ such that $I_i X_i \subseteq M$ for $I = 1, 2$ then $\exists\, J \in \chi$ such that $J(X_1 + X_2) \subseteq (I_1 \cap I_2)(X_1 + X_2) \subseteq M$, then $X_1 + X_2 \in Cl_\chi^E(M)$

Lemma 2.4.4: χ is weak closed under intersection if and only if $Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2) = Cl_\chi^E(M_1 \cap M_2)$ for any N-subgroups M_1, M_2 of N-group E.

Proof: We always have $Cl_\chi^E(M_1 \cap M_2) \subseteq Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2)$, for

$x \in Cl_\chi^E(M_1 \cap M_2) \Rightarrow \exists\, I \in \chi$ such that $Ix \subseteq M_1 \cap M_2 \Rightarrow Ix \subseteq M_1, Ix \subseteq M_2$

$\Rightarrow x \in Cl_\chi^E(M_1), x \in Cl_\chi^E(M_2) \Rightarrow x \in Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2)$.

Otherwise if $x \in Cl_\chi^E(M_1) \cap Cl_\chi^E(M_2)$, there exists I_1, $I_2 \in \chi$, such that $I_i x \subseteq M_i$ then there exists $J \in \chi$ such that $J \leq I_1 \cap I_2$, hence $J_x \subseteq M_1 \cap M_2$ and $Cl_\chi^E(M_1 \cap M_2)$.

Next let I_1, $I_2 \in \chi$, then $1 \in Cl_\chi^N(I_i)$ and then $1 \in Cl_\chi^N(I_1) \cap Cl_\chi^N(I_2) = Cl_\chi^N(I_1 \cap I_2)$ hence there exist $J = J.1 \in \chi$ such that $J \subseteq I_1 \cap I_2$ $\therefore$ χ is weak closed under intersection.

Lemma 2.4.5: If χ is closed under intersection then χ is left N-closed if and only if $Cl_\chi^E(M)$ is a N-subgroup for any N-subgroup $M \subseteq E$ and any left N-group E.

Proof: Let x ∈ Cl_{χ}^{E}(M) and n ∈ N, then there exist I ∈ χ such that Ix ⊆ M, since there exist J ∈ χ such that Jn ⊆ I, we have Jnx ⊆ Ix ⊆ M, hence nx ∈ Cl_{χ}^{E}(M).

Next let I ∈ χ and n ∈ N, then Cl_{χ}^{E} (I) = N, hence n ∈ Cl_{χ}^{E} (I) and there is J ∈ χ , such that Jn ⊆I.

Definition2.4.6: χ is inductive if for any I ∈ χ and any left N-subgroup J ⊇ I , J ∈ χ .

Definition2.4.7: A set of essential N-subgroups is called topological filter if it is closed under intersection , inductive and left closed.

Lemma2.4.8: Let χ be inductive , then the following statements are equivalent:

(1) χ is a topological filter.

(2) Cl_{χ}^{E}(M) is an N-subgroup for any N-subgroup M ⊆ E .

Proof: We only need to show the implication (2) ⟹(1) .

It is obvious that χ is weak closed under intersection and left closed as Cl_{χ}^{E}(M) is an N-subgroup for any N-subgroup M ⊆ E . Then χ is a topological filter because it is inductive, therefore it is closed under intersection.

Definition2.4.9: A topological filter is a linear filter whenever it satisfies: if I ⊆ N and J ∈ $\mathcal{T}$ satisfy (I: y) ∈ $\mathcal{T}$ for any y ∈ J , then I ∈ $\mathcal{T}$.

Proposition2.4.10: Let $\mathcal{T}$ be a topological filter, then the following statements are equivalent:

(c) $\mathcal{T}$ is a linear filter .

(d) $Cl_{\mathcal{T}}^{E}Cl_{\mathcal{T}}^{E} = Cl_{\mathcal{T}}^{E}$ for any N-group E.

Proof: (a) ⟹(b) If $M \subseteq E$ is an ideal and $x \in Cl_{\mathcal{T}}^{E}Cl_{\mathcal{T}}^{E}(M)$, $\exists\, I \in \mathcal{T}$ such that $Ix \subseteq Cl_{\mathcal{T}}^{E}(M)$, then to any $y \in I$ $\exists\, I_y \in \mathcal{T}$ such that $I_y yx \subseteq M$, $I_y y \subseteq (M : x)$ ⟹for any $I_y yx \subseteq M$, hence $((M : x) : Y)$ belongs to $\mathcal{T}$, therefore $(M : x) \in \mathcal{T}$ as $\mathcal{T}$ is a linear filter

$\Rightarrow x \in Cl_{\mathcal{T}}^{E}(M)$

$\therefore\ Cl_{\mathcal{T}}^{E}Cl_{\mathcal{T}}^{E}(M) \subseteq Cl_{\mathcal{T}}^{E}(M)$

$Cl_{\mathcal{T}}^{E}(M) \subseteq Cl_{\mathcal{T}}^{E}Cl_{\mathcal{T}}^{E}(M)$ is obvious.

Thus $Cl_{\mathcal{T}}^{E}(M) = Cl_{\mathcal{T}}^{E}Cl_{\mathcal{T}}^{E}(M)$.

(b) ⟹ (a) Let $J \in \mathcal{T}$ and $I \subseteq N$ be such that for any $y \in J$ the ideal $(I : y) \in \mathcal{T}$,

Hence $N = Cl_{\mathcal{T}}^{N}(J) \subseteq Cl_{\mathcal{T}}^{N}\, Cl_{\mathcal{T}}^{N}(I) = Cl_{\mathcal{T}}^{N}(I)$

$\therefore Cl_{\mathcal{T}}^{N}(I) = N$ and we have $I \in \mathcal{T}$, (when $\mathcal{T}$ set of N-subgroups)

Note. 2.4.11: When $\mathcal{T}$ is only a topological filter, we have that $Cl_{\mathcal{T}}^{E}(M)$ is N- subgroup. When $\mathcal{T}$ is linear filter $Cl_{\mathcal{T}}^{E}(M)$ is $\mathcal{T}$ closed.

Proposition2.4.12: Let $M \subseteq E$ be an ideal then the following statements are equivalent:

(a) M is χ-honest in E.

(b) In addition χ is inductive, then above is equivalent to

(c) For $m \in (Cl_{\chi}^{E}(M) \setminus M$ we have $(M: m) = Ann(m)$

(d) For $m \in (Cl_{\chi}^{E}(M) \setminus M$ we have $Nm \cap N = 0$

Proof: (c) ⟹(d) Lemma 2.2.7

(d) ⟹(a) Lemma 2.2.7

(b) ⇒(c) Let x ∈ $(Cl_{\chi}^{E}(M)\setminus M$, then there exists I ∈ χ with Ix ⊆ M, then Ix = 0

[∵ M is honest in E, x∉M]

Hence I ⊆ Ann (x) ⊆ (M: x), therefore (M: x) ∈ χ [∵ χ is inductive]

Hence (M: x) = Ann (x).

(c)⇒(d) Lemma 2.2.7

Note 2.4.13: We get lemma 2.2.8, corollary 2.2.9, corollary 2.2.10, remark 2.2.11, note 2.2.13

Corollary2.4.14: Let χ be weak closed under intersection and proper essential N-subgroups of N are distributively generated, then any χ-honest N-subgroup M ⊆ E satisfies either M ⊆ T_{χ}(E) or T_{χ}(E) ⊆M.

Proof: Since χ weak closed under intersection and proper essential N-subgroups of N are distributively generated $Cl_{\chi}^{E}(M)$ and T_{χ}(E) are subgroups . The proof follows from 2.2.12.

Note2.4.15: Following it we get corollary 2.2.14

Note2.4.16: For χ as a set of essential N-subgroups of N we get lemma 2.3.10.

Moreover if M is E-closed we get all relations between χ-closed, χ-torsion, torsion, superhonest N-subgroups as in section 3.

Let χ be a non empty set of N-subgroups such that 0∉ χ of near ring N.

For this set also giving the definitions in the same way we get all results of this section.